4 Mathematical Scenarios[5]

In common usage, a scenario is the set up for a story: the main characters, important facts about them and their relationships. *Mathematical scenarios* consist of some unspecified mathematical objects (called the *input objects*) and some basic assumptions about the input objects (called the *assumption* of the scenario). The assumptions are either definite assertions, or indefinite assertions that depend on the input variables. *Mathematical scenarios are the starting point for nearly any investigation or discussion in mathematics.* (Note, however, that the terminology "mathematical scenario" is not standard.)

Here are some simple examples of mathematical scenarios:

Scenario 1.
Input objects: Real numbers x and y
Assumption: $x^2 + y^2 \geq 16$ and $x \leq y$.

Scenario 2.
Input object: A set S of integers
Assumption: There is no integer bigger than 1 that is a divisor of every member of S.

Scenario 3.
Input objects: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a real number t.
Assumption: $f(t) > t$.

A mathematical scenario establishes a situation involving "characters" (certain objects) and assumption (which may be a single assumption, or an "and" of two or more assumptions) about the characters. The assumption is an assertion (definite or indefinite) whose only essential variables are input variables.

If the input variables are assigned values, the assumption becomes either true or false. When we use mathematical scenarios, we view the assumption as a requirement on the assignment of the input variables; the assertion only applies to inputs that satisfy this requirement. A choice of input values that makes the assumption true is said to *satisfy the assumption* or to *satisfy the scenario* and is called a *feasible instance* of the scenario. A choice of input values that makes the assumption false is said to *violate the assumption* or *violates the assumption* and is an *infeasbile instance* of the scenario.

In Scenario 1, for example, $x = 2$ and $y = 5$ is a feasible instance while $x = 2$ and $y = 3$ is an infeasible instance. We can also represent the feasible instance by an ordered pair $(2, 5)$, where we assume that we've fixed the first coordinate to correspond to x and the second to y.

Equivalent scenarios Consider the following scenarios:

Scenario 4.
Input object: Integer n
Assumption: n is prime.

Scenario 5. Input object: Real number n
Assumption: n is a prime integer,

Scenario 6. Input object: Prime integer n
Assumption: None

While they are expressed differently, all of these scenarios are essentially the same. They differ in the initial object type that n is assigned, but when this type is combined with the assumption, they set up the situation that n is a prime integer.

The set of feasible instances of a scenario Suppose we have a scenario with a single input variable x from a set T and the assumption $A(x)$.

$$\{x \in T : H(x)\},$$

which is read "The set of x in T satisfying $H(x)$", is the set of all feasible instances of the scenario.

If there is more than one input variable in the scenario, each feasible instance is a list of values (of length equal to the number of variables).'

When we introduced sets earlier, we pointed out that there is a difficulty in describing sets with a large number or infinitely many mambers. The above notation provides us with a very useful way for describing a huge variety of sets. Here are a few examples.

- $\{k \in \mathbb{Z} : k-1 \text{ is divisible by } 3\}$. This set consists of infinitely may integers including -2, 1 and 4.
- $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$. This is the set of ordered pairs of real numbers. If we graph this set in the x-y plane we get a circle of radius 2 centered at the origin.
- $\{x, A\} \in \mathbb{Z} \times \mathcal{P}_{\mathbf{fin}}(\mathbb{Z}) : x \in A\}$. This is the set of pairs (x, A) where x is an integer, A is a finite subset of integers and $x \in A$. For example $(7, \{-3, 7, 10\})$ is in this set but $(0, \{1, 2, 3, 4, 5\})$ is not.

Remark 4.1. (Understanding the definition of a set.) The notation for each of these sets, especially the third, is a bit tricky to understand. One source of difficulty in the third example is simply to understand the object type of the members of the set. In this case, each member is an ordered pair whose first entry is an integers and whose second entry is a finite set of integers. In general, when presented with a new set, the first question to ask is: what type of objects are the members of the set. Having done this, the mathematician gains further understanding by finding one or more examples of a member of the set, and one or more examples of objects of the same type as the members of the set.

In each of these examples, the set represents the set of feasible solutions to the mathematical scenario whose input object(s) are listed to the left of the colon, and whose assumption is given to the right of the colon.

We have the following terminology for scenarios:

- A scenario is *feasible* if it has at least one instance, which means that the set of feasible instances is nonempty.

- A scenario is *uniquely feasible* if it has exactly one feasible instance.
- A scenario is *infeasible*, *contradictory* or *impossible* if it has no feasiblle instances so that the set of feasible instances is empty.

Example 4.1. Consider the following three scenarios with two input objects, both of which are real numbers x and y.

Scenario A has one assumption $x + y = 3$. This scenario is feasible since, for example, $x = \frac{1}{2}$ and $y = \frac{5}{2}$ is an instance. It is not uniquely feasible since it has other instances also.

Scenario B has the same requirement $x + y = 3$, and also the additional requirements that x and y are integers, $y > x$ and $x > 0$. This scenario is uniquely feasible since $x = 1$ and $y = 2$ is the only instance.

Scenario C has the assumption $x + y = 3$ and $x^2 + y^2 \geq 5$. This scenario is infeasible-there is no way to choose real numbers x and y to satisfy both conditions.

Mathematical scenarios play a central role in thinking and communicating about mathematics: Many mathematical problems begin by describing some scenario, and then ask you to find a feasible solutions, or describe all feasible solutions. As we are about to see, any existential or universal assertion concerns a mathematical scenario, and as we'll see later, the notion of a mathematical scenario underlies mathematical proofs.

Scenarios and Existential assertions An existential assertion has the form: "There exists an object x of type T satisfying $P(x)$". This principle is associated with the scenario whose input object is of type T and is represented by x, and whose assumption is $P(x)$. The existential principle simply makes the claim that this associated scenario is feasible.

Scenarios and unique existential assertions A unique existential assertion has the form: "There exists a unique object x of type T satisfying $P(x)$". This principle is associated with the scenario whose input object is of type T and is represented by x, and whose assumption is $P(x)$. The unique existential principle simply makes the claim that this associated scenario has exactly one feasiable solutions.

Scenarios and universal assertions A universal assertion has the form: "For any object x of type T that satisfies $A(x)$, we must have $C(x)$.' We can associate the principle to the mathematical scenario with input object x and assumption $A(x)$, which we refer to as the *hypothesis* of the universal assertion. The principle says that any feasible instance of the hypothesis must satisfy $C(x)$.

Let's analyze some previously stated universal assertions from this point of view. For Universal Principle 2.5 we have:

Input. Positive integers a and b

Assumption. a is a positive integer, b is a positive integer and b is prime.

Conclusion. b is a divisor of $a^b - a$.

For Universal Principle 2.7, we have:

Input. The sets A, B and C

Assumption. $A \neq B$.

Conclusion. $A \cup C \neq B \cup C$ or $A \cap C \neq B \cap C$.

The following terminology is helpful in formulating what it means for a universal assertion to be true.

1. A *test case* of a universal assertion is a *feasible instance* of the associated mathematical scenario.
2. A *successful test case* of a universal assertion is a test cqse that makes conclusion true.
3. A *counterexample* or *unsuccessful test case* for a universal assertion is an instance for which the conclusion is false.

For Universal Principle 2.5, we have:

- The choice $a = 8$ and $b = 3$ is a test case since it satisfies the assumption. It is also a successful test case because it also satisfies the conclusion, since 3 is a divisor of $8^3 - 8 = 504$.
- Setting $a = 5$ and $b = 4$ is not a test case because it does not satisfy the assumption that b is prime. Since it is not a test case, it is neither a successful test case or a counterexample.

In general notice that:

- Every test case is either a successful test case or a counterexample but not both.
- An assignment of the input variables that makes the assumpton *false* is not a test case, and so can not be either a successful test case or a counterexmample.

Using this terminology we can say:

> A universal principle is a universal assertion for which every test case is successful, or equivalently, the assertion has no counterexamples.

Here's another example. Consider the following two assertions:

Assertion D. Every prime number is odd.

Assertion E. There is no largest prime number.

For assertion D:

- The choice $k = 11$ is a test case (since 11 is prime) and is a successful test case (since 11 is odd).
- The choice $k = 15$ is not a test case since 15 is not prime.
- The choice $k = 2$ is a test case (since 2 is prime), and is a counterexample since 2 is not odd.

Since Assertion D has a counterexample, it is *not* a universal principle.

Assertion E does not look like a universal assertion but it turns out that it is a universal assertion in disguise. To formulate this as a universal assertion, observe that Assertion B has the following meaning: If you give me any prime number, I can give you a larger one. In other words, in the scenario where n is a prime number we want to conclude that there is a prime number m that is larger than n. So Assertion E is equivalent to:

Assertion E′. For every prime number n there is a larger prime number.

Here are some successful test cases:

- Choose $n = 3$. Then n is prime and 7 is a larger prime number. (Notice we have many other choices besides 7.)
- Choose $n = 17$. Then 37 is a larger prime number.

How about $n = 12553$. For one ting it's not clear whether 12553 is a test case, which in this case requires that it be prime. If it is a test case, then to be successful we'd need a larger prime number.

As usual with a universal assertion, even if we check a few successful test cases, we can't be sure that the assertion is true. Later we'll see that this universal assertion is indeed true (and so is a universal proposition). In fact it is one of the most famous (and oldest) universal propositions known.

Vacuously true universal assertions. For a universal assertion of the form "for all x that satisfy $A(x)$, we have $C(x)$" we saw that this assertion is true provided that every test case is successful. What if there are no test cases at all? Can this happen?

It certainly can happen. For example, consider the universal assertion: For any real number x, if $x^2 < -1$ then $x \geq 1000$. Notice that there are no real numbers that satisfy the assumption, and therefore there are no test cases.

Now, in this case is the universal assertion true or false. Such a universal assertion is considered to be true since there are no counterexamples, and a universal assertion with no counterexamples is, by definition, true.