3 Communicating mathematics I: Assertions [4]

To communicate effectively about mathematics you need all of the basic skills needed to communicate about any subject. Mathematical ideas are expressed using full sentences in English (or whatever language we're communicating in), and communication follows the basic rules of grammar and logic. Mathematics (like any subject) has its own specialized terminology, and that terminology must be used correctly. There are three key differences between mathematical communication and communication in other fields:

- Mathematics has considerably more *logical complexity* than most other subjects, and is much less tolerant of ambiguity than most other subjects. In mathematics, you must write/say exactly what you mean, and you must make sure that your exact meaning is clear to the reader/listener.
- Mathematics deals with precisely defined objects of various ty;es, and the way you can manipulate and analyze an object depends very strictly on its type. It is crucial that objects of each type are treated appropriately, for example, you can take the derivative of a function from the real numbers to the real numbers, but you can't take the derivative of a function from the integers to the integers, and you can't take the derivative of a set.
- Mathematics makes extensive use of *variables*, which are symbols (typically letters) that represent unspecified mathematical objects. The proper use of variables is one of the most important (and, for beginners, challenging) skills required for communicating mathematics.

All three of these aspects are present in other fields, but in mathematics they are critical. The most common communication errors arise when the communicator does not properly respect these three aspects of mathematical communication If the first requirement is not properly dealt with, the communication will contain *errors of logic, or ambiguities.* If the second requirement is not properly dealt with, you make an *error of type* or *error of category* in which you use an object in a way inappropriate to its type, and if you fail to meet the third requirement you are guilty of *misuse of variables.* Any of these errors can completely undermine your ability to communicate mathematics effectively.

In this section, we'll start building a foundation for communicating about mathematics by idenfitying and distinguishing different types of sentences that are used in mathematical communication. We have already introduced universal principles. Universal principles belong to the more general class of sentences called *assertions*, which are sentences that make a mathematical claim. Assertions are classified into *definite assertions* and *indefinite assertions.* Accurate communication about mathematics requires a clear understanding of these types and the relationships between them.

[4] Version 2-2-2015.

Remark 3.1. There is no general agreement among mathematicians what to call these different types of sentences. What we call "definite assertions" here are called "propositions" or "statements" in other books, and "indefinite assertions" are sometimes called "predicates" or "open sentences".

3.1 Definite Assertions

Assertions are a type of mathematical sentence that makes a claim. We classify assertions into *definite assertions* and *indefinite assertions*. We'll start by discussing definite assertions.

A definite assertion makes a claim that is true or false. We may not know whether the claim is true or false, but we know from the form of the sentence that it is one or the other. For example, the sentence:

1. 12345678900100987654321 is a prime number,

is either true or false because every number is either prime or not prime. You may not know whether the sentence is true or not (I don't), but it is still a definite assertion.

The labels **T** and **F** are called truth values. Every assertion has exactly one truth value. For an assertion A, we write $TV(A)$ for its truth value.

Here are some more examples of definite assertions.

2. 58 multiplied by 49 is greater than or equal to 56 multiplied by 51.
3. $58 \times 49 \geq 56 \times 51$.
4. The triangle having vertices $(0, 3)$, $(2, 6)$ and $(5, 4)$ has area 8.
5. For every real number x, $x^2 + 110 \geq 21x$.
6. There is a real number x such that $x^2 + 110 \geq 21x$.
7. There is no prime number greater than 10^{100}.
8. There is a real number b so that for any real number x that is bigger than b, $2^x \geq 100^x$.

Assertions 6 and 8 are true, while Assertions 2, 3, 4, 5 and 7 are all false. For some of these statements it is easy to determine whether or not its true, but for some it may not be clear why its true or false.

Notice that assertions 2 and 3 have *exactly the same meaning*; the first is expressed using only English words and numbers, while the second sentence is an abbreviation of the first using mathematical symbols.

Assertion 5 is making a series of claims, one for each real number x. Such a claim is called a *universal assertion*. This one happens to be false; to show this we just need to find a single value for x that makes this false. You can check that $x = 10.5$ makes this false.

A universal assertion that is true is called a *universal principle*. In Section **??**, we presented several universal principles.

Assertion 6 has a similar structure to Assertion 5 but starts out with “There is a positive real number x” instead of “For every real number x”’. This type of assertion is called an *existential assertion*. For it to be true, we only need that there is at least one choice of x to make the condition true. Since it is possible to select some x to make $x^2 + 110 \geq 21x$ true, for example $x = 0$, the assertion is true.

In contrast to the sentences above, here are some examples of mathematical sentences that are not assertions.

9. Is 77 a prime number?

10. How many positive factors does 120 have?

11. We say that a set of numbers is *bounded above* if there is a number that is bigger than every number in the set.

12. Compute the product of the smallest 10 integers.

Sentence 9 and 10 are questions, sentence 12 is an *imperative* sentence, that *instructs*, *commands* or *asks* the reader to do something, and sentence 11 is a *mathematical definition*, which *informs* the reader what the writer means when he says that a set of numbers is bounded.

Remark 3.2. There are logical difficulties in drawing an exact line between what is, and what is not, a mathematical assertion. The sentence ”This statement is not true” looks like an assertion because it makes a specific claim. However it is not an assertion because it is neither true nor false: it is not true (because if it were true then it would have to be not true) and it can’t be false (since if it were false it would have to be true). There are sentences in mathematics that are even more difficult to deal with than this one and these sentences play a major role in the field of *Mathematical Logic*. Fortunately, at the level of this course one rarely runs into these difficult sentences (unless you’re looking for them), and we’ll simply avoid these logically troublesome sentences.

Now consider the following sentence:

13. $n - 1$ is divisible by 4.

This sentence seems to be making a claim, but it is not a definite assertion, because the meaning of the sentence depends on the unspecified object n. If $n = 9$ it becomes a true assertion, whle for $n = 15$ it becomes a false assertion. This kind of sentence is called an *indefinite assertion* because it contains unspecified variable(s), and sutstituting different vallues for these variables results in sentences with different meanings, and possibly different truth values.

Indefinite assertions are a central part of mathematical communication, and we’ll discuss them in more detail in Section 3.2.

3.2 Indefinite assertions, universal assertions, and existential assertions

So far the assertions we've discussed have been definite assertions. Now we'll introduce indefinite assertions. Let's start with a non-mathematical example. Consider the sentence:

14. Maria is a chemistry major.

Is this a definite assertion? It looks like it, but there's a difficulty. Is this sentence true or false? Let's assume we're told that Maria is a student at Rutgers. There are many students named Maria at Rutgers. Which one is the sentence talking about? Probably some of the Marias are chemistry majors and some of them aren't. The uncertainty about which Maria the sentence refers makes the meaning of the sentence uncertain. If we knew which Maria was meant, we'd be able to tell whether the sentence is true or false. A sentence of this type is called an *indefinite assertion*.

Now, let's look at two other sentences built from this one:

15. There is a student named Maria who is a chemistry major.

16. Every student named Maria is a chemistry major.

These two sentences are definite assertions: they are each true or false. For the first to be true we need that at least one of the students named Maria is a chemistry major, otherwise its false. For the second to be true, every student named Maria must be a chemistry major, otherwise it's false. The first sentence is an existential assertion and the second is a universal assertion.

There is an analogous situation for mathematical sentences. Let's take a look at a sentence that is related to both Assertion 5 and Assertion 6:

17. $x^2 + 110 \geq 21x$.

The meaning of this sentence depends on what x stands for. Each value of x gives a different meaning to the sentence, and some may be true and some false.

This sentence has the following properties:

- It contains a variable.
- When we substitute the variable by a permitted value, we get a meaningful sentence and that sentence is true or false, though the truth or falsity may change with x.

Such a sentence is called an *indefinite assertion*.

Notice that the universal assertion 6 and the existential assertion 5 are definite assertions that are built from this indefinite assertion.

The variable x in assertion 17 is said to be a *free variable*. This sentence has a different interpretation for different values of the variable x and it is this reason why the sentence is an indefinite assertion.

Once we modify the sentence to assertion 6 or to assertion 5, the variable is said to be a *bound variable* or a *dummy variable*. For either of these sentences the sentence becomes strange if we substitute a specific object for x. For example, if you subsitute $x = 6$ into Sentence 6 you get: "For all real numbers 6, $6^2 + 110 \geq 21 \times 6$'; here the peculiar phrase "For all real numbers 6" is a clue that this substitution is not permitted because x is a dummy variable.

Dummy variables also have the property that they are *inessential*: You can express the same assertion 6 without mentioning any variable: Every real number has the property that if you square it and add 110 the result is larger than or equal to 21 times that number. Using a variable makes it easier to express the sentence, but the variable is not required to express the sentence. You can replace all occurences of x by a different variable and the meaning of the sentence will remain the same. (In section **??** where we discuss proper use of variables in detail, we'll give other examples of the use of dummy variables such as the index of summation i in $\sum_{i=1}^{1} 0i^2$ is a dummy variable and the variable x in the set definition $\{x \in \mathbb{R} : x^4 - x^2 \leq 1\}$.)

The modification of an indefinite assertion with free variable x by adding "For all x" in front is called *universal quantification* and the modification by adding "There exists x" is called *existential quantification*.

In discussing assertions, it is common to represent an assertion by a letter such as A. If A is an indefinite assertion involving x we may indicate this by writing $A(x)$. In this symbolic form, the assertion "There exists x such that $A(x)$" is abbreviated by "$\exists x, A(x)$'; the symbol $\exists$ is called the *existential quantifier*. The assertion "For all x, $A(x)$" is symbolically abbreviated by "$\forall x, A(x)$". The symbol $\forall$ is called the *universal quantifier*.

Universal and existential quantification is closely connected to the *truth set* of an assertion. Let $P(x)$ be any indefinite assertion involving the variable x, where x stands for an object of a specific type T (such as "real number" or "set of integers"). The *truth set of* $P(x)$ is the set of those objects from T such that if you replace x by the object $P(x)$ becomes true. This set is denoted by

$$\{x \in T : P(x)\},$$

which is read: "The set o x in T that satisfy $P(x)$".

We can build two definite assertions from $P(x)$:

Existential assertion. There exists an $x \in T$ such that $P(x)$.

Universal assertion. For all $x \in T$, $P(x)$ holds.

The first assertion can be reformulated as:" The truth set of $P(x)$ is nonempty", while the second can be reformulated as "The truth set of $P(x)$ is all of T'".

.

Here is an important variant of an existential assertion:

Unique existential assertion. There is exactly one x in T such that $P(x)$.

This kind of assertion is called an *existence and uniqueness theorem.* It is really two assertions in one: "There exists an x in T such that $P(x)$" (which is just the usual existential asertion) and "There is at most one $x \in T$ such that $P(x)$". This second assertion may look like an existential assertion but it isn't; the words "at most one" completely changes the meaning. The assertion"there is at most one $x \in T$ such that $P(x)$' can be restated as "Whenever x and y are members of T such that both $P(x)$ and $P(y)$ are true, we must have $x = y$" and this is actually a universal assertion: "For all $x \in T$ and $y \in T$, if $P(x)$ and $P(y)$ then $x = y$."

The phrases "There exists $x \in T$", "For all $x \in T$" and "There exists a unique x in T" have the following symbolic abbreviatons:

$\forall x \in T, P(x)$ abbreviates "For all $x \in T$, P $P(x)$ holds."

$\exists x \in T, P(x)$ abbreviates "There exists $x \in T$ such that $P(x)$."

$\exists! x \in T$ abbreviates "There exists a unique $x \in T$ such that $P(x)$."

If the set T is understtood from context we can write simply $\forall x, P(x)$, $\exists x, P(x)$, or $\exists! x, P(x)$.

Assertions with multiple variables. Assertions can have more than one variable. Each variable in the assertion is either free or dummy, and it's important to distinguish these. The assertion is definite if it has no free variables, otherwise it is indefinite. Here are some examples; in these sentences x and y are real number variables.

18. $x+y^2 \geq xy$. This has both x and y as free variables. It is an indefinite assertion depending on both x and y. We can abbreviate it symbolically by $B(x, y)$.

19. For all real numbers x, $x+y^2 \geq xy$. This has x as a bound variable and y as a free variable. It is an indefinite assertion depending on y (but not x) and is denoted symbolically by $\forall x \in \mathbb{R}, B(x, y)$. Since x is a bound variable we can reformulate the sentence without mentioning x: For every real number, the sum of the number and y^2 is at least the product of the number and y. We can't reformulate the sentence without y.

20. There exists a real number y such that $x + y^2 \geq xy$. This has y as a bound variable and x as a free variable. It is an indefinite assertion depending on x (but not y) and is denoted symbolically by $\exists y \in \mathbb{R}, B(x, y)$. Since y is a bond variable we can reformulate the sentence without mentioning y: There is a real number with the property that the sum of x and the square of the chosen number is at least x times the chosen number.

21. For all real numbers x, there exists a real number y such that $x + y^2 \geq xy$. This has both x and y as bound variables, and no free variables, and so is a definite assertion. It is denoted symbolically by $\forall x \in \mathbb{R} \exists y \in \mathbb{R}, B(x, y)$. Since x and y are both bound variables, we can reformulate this sentence without either x or y: For any real number, it is possible to choose a seoncd number so that the first number plus the square of the second is at least the product of the two numbers.

22. There exists a real number y such that for all real numbers x, $x + y^2 \geq xy$. This has both x and y as bound variables, and no free variables, and so is a definite assertion. It is denoted symbolically by $\exists y \in \mathbb{R} \forall y \in \mathbb{R}, B(x, y)$. Since x and y are both bound variables, we can reformulate this sentence without either x or y: There is a real number, so that for any real number, For any real number, the first number plus the square of the second is at least the product of the two numbers.

Sentences 21 and 22 are examples of extremely common and important classes of assertions. Let's understand what each one means, and compare them.

Sentence 21 says: No matter what real number x is chosen, we can choose y so that $x + y^2 \geq xy$. Is this true? Let's try an example If you choose $x = 10$ then it's easy to see that many values of y works, for example $y = 20$. If you choose $x = -5$ then again many choices of y work, for example $y = 5$ works. Notice that the value of y chosen may depend on x. It is easy to argue that for x negative we can take $y = 1$, since then the lefthand side is $x + y^2 = x + 1$ and the righthand side is $x(1) = x$, and so the lefthand side is larger than the righthand side. If instead $x \geq 0$ we just take $y = x$ and then the lefthand side is $x + x^2$ which is certainly at least the righthand side which is x^2.

Sentence 22 says something different. It says that it is possible to select a single y that works for all x. This turns out to be impossible. For example, if you try $y = 100$, it doesn't work for $x = 200$, and no matter which y you pick, there will be an x for which $B(x, y)$ is false.

A sentence whose form is like that of 21 is called a "for all-there exists" or $\forall\exists$ type of sentence, while a sentence whose form is like that of 22 is called a "there exists-for all" or $\exists\forall$ sentence.

We just saw that it is possible for the $\forall\exists$ sentence to be true while the $\exists\forall$ sentence is false. However, this can't happen the other way around. If a sentence $\exists y \forall x C(x, y)$ is true then the sentence $\forall x \exists y C(x, y)$ must also be true. To see this, assume that $\exists y \forall x C(x, y)$ is true. Then it is possible to pick y to be a fixed value, call it y_0 so that for every choice of x $C(x, y_0)$ holds. Now to verify that $\forall x \exists y C(x, y)$ we need to show that given any x we can choose a y that makes $C(x, y)$ true. So for any selected x we'll choose y to be y_0. We know $C(x, y_0)$ holds.

Earlier we defined the truth set of an assertion $A(x)$ with free variable x as the set of values of x that make $A(x)$ true. We denote this by $\{x : A(x)\}$.

We can do the same with assertions that have more than one free variable. If $A(x, y)$ is an assertion with free variables x, y then

$\{(x, y) : A(x, y)\}$ is the set of ordered pairs that, when substituted for x, y make $A(x, y)$ true.

We can also form the set $S = \{x : A(x, y)\}$, which is the set of values of x that make $A(x, y)$ true. Notice that the values of x that make $A(x, y)$ true depends on y, so $\{x : A(x, y)\}$ is a set whose members depend on what y is. For this reason it is customary to write $S(y)$ or S_y for the name of the set (rather than just S to emphasize that the set varies depending on y.

Example 3.1. Suppose that $A(x, y)$ is the indefinite assertion $y \geq x^2$. Then the set $S(y) = \{x : x^2 \leq y\}$ is a set that varies with y. It is empty if $y < 0$ and is equal to the set of real numbers between $-\sqrt{y}$ and $\sqrt{y}$ if $y \geq 0$.