Intro to Mathematical Reasoning (Math 300) – Homework 3[1]

1. Prove: "For all positive real numbers x and y, $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$.
2. One of the definitions that the book gives for "odd" is that an integer a is odd provided that there is an integer b such that $a = 2b + 1$. Use this definition to prove: "For all odd integers p and odd integers q, their product pq is odd."
3. In supplement 1, we gave the following definition: for all integers m and n, we say that m *is a divisor of* n provided that there is an integer d such that $dm = n$.

   Consider the following statement: "For all integers r, s, t, if r is a divisor of s and r is a divisor of t then $2r$ is a divisor of $s + t$." Here is a faulty proof:

   > **Proof.** We use the arbitrary value method. Let k, m, and n be arbitrary integers such that k is a divisor of m and k is a divisor of n. We will show that $2k$ is a divisor of $m + n$.
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   > Since k is divisor of m, there is an integer d such that $dk = m$. Since k is divisor of n, there is an integer d such that $dk = n$. Then $m + n = dk + dk = 2dk$. Since d is an integer, we conclude that $2k$ is a divisor of $m + n$.
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   > Since k,m and n are arbitrary integers satisfying that k is divisor of m and k is a divisor of n we conclude that for all integers r, s, t if r is a divisor of s and r is a divisor of t then $2r$ is a divisor of $s + t$.

   Use "test by trial value" (see supplement 3) to uncover the flaw in the proof and carefully explain the flaw.
4. Prove: For all integers p, q, r, s, t, if p is a divisor of q and p is a divisor of r then p is a divisor of $qs + rt$.

   After proving it, go through the steps of your proof on the trial values $p = 7$, $q = 14$, $r = 21$, $s = -3$, $t = 8$.
5. Prove: for all real numbers u, v if $u < v$ then there exists a real number w such that $u < w < v$. (Hint: See the example in supplement 3, page 8.)
6. Prove: for all positive real numbers u, v if $u^2 < v$ there exists a real number w satisfying $u < w$ and $w^2 < v$. (Hint: See the example in supplement 3, page 9.)
7. A *consecutive list of integers* is a finite list of integers where each integer in the list is one more than the previous one. The *length* of such a list, is the number of integers in the list. Prove: for every nonzero integer n there is a consecutive list having even length whose members sum to n.
8. Let us say an integer n is a *nontrivial consecutive sum* provided that there is a consecutive list of positive integers of length at least 2, whose sum is n. Investigate the following problem:

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Is there a simple way to tell which positive integers are nontrivial consecutive sums? Your goal to make a conjecture of the form "For all integers n, n is a nontrivial consecutive sum if and only if $P(n)$" where $P(n)$ is a predicate that is very easy to check. (Hint: try some examples and look for patterns!)

Remember that a conjecture is a statement that you believe to be true but have not proved. You are not expected to prove your conjecture. You should, however, explain the reason that you made your conjecture, and present whatever evidence you have that the conjecture is true. (If you are able to prove the conjecture, this will be very impressive and will earn extra credit.)