4 Communicating mathematics I: Assertions [5]

To communicate effectively about mathematics requires all of the basic skills needed to communicate about any subject, and more. Mathematical ideas are expressed using full sentences in English (or whatever language we're communicating in), and communication follows the basic rules of grammar and logic. Mathematics (like any subject) has its own specialized terminology, and that terminology must be used correctly.

There are three key differences between mathematical communication and communication in other fields:

- Mathematics has considerably more *logical complexity* than most other subjects, and is much less tolerant of ambiguity than most other subjects. In mathematics, you must write/say exactly what you mean, and you must make sure that your exact meaning is clear to the reader/listener.
- Mathematics deals with precisely defined objects of various types, and the way you can manipulate and analyze an object depends very strictly on its type. It is crucial that objects of each type are treated appropriately, for example, you can take the derivative of a function from the real numbers to the real numbers, but you can't take the derivative of a function from the integers to the integers, and you can't take the derivative of a set.
- Mathematics makes extensive use of *variables*, which are symbols (typically letters) that represent unspecified mathematical objects. The proper use of variables is one of the most important (and, for beginners, challenging) skills required for communicating mathematics.

These aspects are present in other fields, but in mathematics they are much more important. The most common communication errors in mathematics arise when the communicator does not properly respect these three aspects of mathematical communication. If the first requirement is not properly dealt with, the communication will contain *errors of logic, or ambiguities.* If the second requirement is not properly dealt with, you make an *error of type* or *error of category* in which you use an object in a way inappropriate to its type, and if you fail to meet the third requirement you are guilty of *misuse of variables.* Any of these errors can completely derail your ability to communicate mathematics meaningfully and effectively.

In this section, we'll start building a foundation for communicating about mathematics by idenfitying and distinguishing different types of sentences that are used in mathematical communication. We have already introduced universal principles. Universal principles belong to the more general class of sentences called *assertions*, which are sentences that make a mathematical claim. Assertions are classified into *definite assertions* and *indefinite assertions.* Accurate communication about mathematics requires a clear understanding of these types and the relationships between them.

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Remark 4.1. There is no general agreement among mathematicians what to call these different types of sentences. What we call "definite assertions" here are called "propositions" or "statements" in other books, and "indefinite assertions" are sometimes called "predicates" or "open sentences".

Assertions as mathematical objects Assertions are a central part of our tool kit for studying the mathematical universe. While we usually think of assertions as standing outside and separate from the mathematical universe (in the same way that one thinks of chemistry lab equipment as separate from the materials that it being used to study), assertions have lots of common features with mathematical objects. We'll see that there are operators that change an assertion into a different assertion, and operations for combining two assertions into one. We'll have methods for determining whether two assertions are "equivalent", that is, have the same meaning. We will also represent assertions by variables (usually upper case letters).

The field of mathematical logic takes this similarity one step further and treats mathematical assertions as just another mathematical object living in the mathematical universe. (Just as we could view chemistry lab equipment as consisting of chemical materials that we can study just like any other materials). While the mathematical logic approach is fascinating and important, for most purposes mathematicians normally think of assertions and other tools for studying the mathematical universe as sitting outside the universe. This is what we do in this course.

4.1 Definite Assertions

Assertions

Assertions are a type of mathematical sentence that makes a claim. We classify assertions into *definite assertions* and *indefinite assertions*. We'll start by discussing definite assertions.

A definite assertion makes a claim that is true or false. We may not know whether the claim is true or false, but we know from the form of the sentence that it is one or the other. For example, the sentence:

1. 12345678900100987654321 is a prime number,

is either true or false because every number is either prime or not prime. You may not know whether the sentence is true or not (I don't), but it is still a definite assertion.

We often abbreviate true by **T** and false by **F**. **T** and **F** are called truth values. Every definite assertion has exactly one truth value. For a definite assertion A, we write $TV(A)$ for its truth value.

Here are some more examples of definite assertions.

2. 58 multiplied by 49 is greater than or equal to 56 multiplied by 51.

3. $58 \times 49 \geq 56 \times 51$.

4. The triangle having vertices $(0, 3)$, $(2, 6)$ and $(5, 4)$ has area 8.

5. For every real number x, $x^2 + 110 \geq 21x$.

6. There is a real number x such that $x^2 + 110 \geq 21x$.

7. There is no prime number greater than 10^{100}.

8. There is a real number b so that for any real number x that is bigger than b, $2^x \geq 100^x$.

9. There is an even number bigger than 4 that can not be expressed as the sum of two prime numbers.

10. Every set of positive real numbers has a least member.

11. Every set of positive integers has a least member.

Assertions 6, 8 and 11 are true, while Assertions 2, 3, 4, 5, 7 and 10 are all false. For some of these statements it is easy to determine whether or not its true, but for some it may not be clear why its true or false.

No one knows whether assertion **??** is true or false. Determining which is a very famous unsolved problem in mathematics.

Notice that assertions 2 and 3 have *exactly the same meaning*; the first is expressed using only English words and numbers, while the second sentence is an abbreviation of the first using mathematical symbols.

Assertion 5 is making a collection of claims, one for each real number x. Such a claim is called a *universal assertion.* This univeral assertion happens to be false; to show this we just need to find a single value for x that makes this false. You can check that $x = 10.5$ makes this false.

A universal assertion that is true is called a *universal principle.* In Section **??**, we presented several universal principles.

Assertion 6 has a similar structure to Assertion 5 but starts out with "There is a positive real number x" instead of "For every real number x"'. This type of assertion is called an *existential assertion.* For it to be true, we only need that there is at least one choice of x to make the condition true. Since it is possible to select some x to make $x^2 + 110 \geq 21x$ true, for example $x = 0$, the assertion is true.

In contrast to the sentences above, here are some examples of mathematical sentences that are not assertions.

12. Is 77 a prime number?

13. How many positive factors does 120 have?

14. We say that a set of numbers is *bounded above* if there is a number that is bigger than every number in the set.

15. Compute the product of the smallest 10 integers.

Sentence 12 and 13 are questions, sentence 15 is an *imperative* sentence, that *instructs*, *commands* or *asks* the reader to do something, and sentence 14 is a *mathematical definition*, which *informs* the reader what the writer means when he says that a set of numbers is bounded.

Remark 4.2. There are logical difficulties in drawing a line between what is, and what is not, a mathematical assertion. The sentence "This statement is not true" looks like an assertion because it makes a specific claim. However it is not an assertion because it is neither true nor false: it is not true (because if it were true then it would have to be not true) and it can't be false (since if it were false it would have to be true). There are sentences in mathematics that are even more difficult to deal with than this one and these sentences play a major role in the field of *Mathematical Logic*. Fortunately, at the level of this course one rarely runs into these difficult sentences (unless you're specifically looking for them), and we'll simply avoid these logically troublesome sentences.

Indefinite assertions, universal assertions, and existential assertions

So far the assertions we've discussed have been definite assertions. Now we'll introduce indefinite assertions.

Let's take a look at a sentence that is related to both Assertion 5 and Assertion 6:

16. $x^2 + 110 \geq 21x$.

This sentence has the following properties:

- It contains a variable that represents an unspecified object of some type. (In this case, one can deduce from the form of the sentence that x represents a real number.)
- The sentence makes a claim about the unspecified object.
- If we replace every occurence of the variable by a single object of the appropriate type. we get a meaningful assertion, which is either true or false.

Such a sentence is called an *indefinite assertion*, and here the variable x is said to be a *free variable* within the sentence.

The universal assertion 6 and the existential assertion 5 are definite assertions that are built from this indefinite assertion by prefacing the sentence with "For all real numbers x" or "There exists a real number x". The phrases "For all" and "There exists" are called "logical quantifiers" or simply "quantifiers". A variable that appears in a sentence with one of these phrases is said to be "quantified".

The variable x in assertion 16 is said to be a *free variable*. Once we modify the sentence to assertion 6 or to assertion 5, by quantifying the variable, the variable is said to be a *dummy variable* (sometimes called a *bound variable*.

Remark 4.3. *Free variables versus dummy variables.* Variables are an essential part of the language of mathematics, and a later section of these notes will focus on the proper use of variables. One crucial aspect to using variables is understanding the distinction between *free* *variables* and *dummy variables*. Whether a variable is a free or dummy variable depends on the way it is used. As just mentioned, a variable that is quantified in a sentence is a dummy variable, but there are other situations where a variable is a dummy. The distinction between free and dummy variables in general can be somewhat hard to understand, and we postpone a thorough discussion of the distinction until the section on use of variables.

Here is the key distinction: When a variable is free in an assertion, that variable represents an object which is essential to the meaning of the sentence. When x is a free variable, it represents a particular, though possibly unknown, object. If y is another free variable, x and y may or may not be the same. When we make the assertion "$x^2 + 110 \geq 21x$" we are saying something about the particular object x. If we change "x" to "y" we have a new sentence which is similar to the first but has a very different meaning. The first sentence tells us something about x, while the second tells us something about y.

In contrast, in a sentence in which x is quantified, x does not represent a particular object. If we replace x by y in 5 we get "For all real numbers y, $y^2 + 110 \geq 21y$" which has the exact same meaning as the first. In fact, we can express the meaning of the sentence without using any variable at all: Every real number has the property that if you square it and add 110 the result is larger than or equal to 21 times that number. Using a variable makes it easier to express the sentence, but the variable is not required to express the sentence.

The modification of an indefinite assertion with free variable x by adding "For all x" in front is called *universal quantification* and the modification by adding "There exists x" is called *existential quantification*.

Symbolic abbreviation of existential and universal statemements If A is an indefinite assertion involving x we may indicate this by writing $A(x)$. In this symbolic form, the assertion "There exists x such that $A(x)$" is abbreviated by "$\exists x, A(x)$'; the symbol $\exists$ is called the *existential quantifier*. The assertion "For all x, $A(x)$" is symbolically abbreviated by "$\forall x, A(x)$". The symbol $\forall$ is called the *universal quantifier*.

Usually, the object x in the indefinite assertion $A(x)$ is restricted to objects of some specific type T. The type T (such as *real number* or *list of integers*) may be understood from context. To make it explicit we often modify the above notation and write:

"There exists an $x \in T$ such that $A(x)$" which is abbreviated by "$\exists x \in T, A(x)$".
"For all $x \in T$ we have $A(x)$" which is appreviated by "$\forall x \in T, A(x)$".

The solution set of an indefinite assertion Universal and existential quantification is closely connected to the *truth set* of an assertion. Let $P(x)$ be any indefinite assertion involving the variable x, where x stands for an object of a specific type T (such as "real number" or "set of integers"). The set of objects in T that satisfy $P(x)$ can be represented using constraint specification:

$$\{x \in T : P(x)\}$$

This is the *solution set* or *truth set* of the indefinite assertion $P(x)$. The statements "$\forall x \in T, P(x)$" and "$\exists x \in T, P(x)$" have a very clear meaning in terms of the solution set of $P(x)$:

"$\forall x \in T, P(x)$" means the solution set of $P(x)$ is all of T.

"$\exists x \in T, P(x)$" means the solution set of $P(x)$ is not the empty set.

Here is an important variant of an existential assertion:

Unique existential assertion. There is exactly one x in T such that $P(x)$. In other words, the requirement $P(x)$ has a unique solution.

This sentence actually combines two assertions in one:

- There exists an x in T such that $P(x)$ (which is the usual existential assertion).
- There is at most one $x \in T$ such that $P(x)$".

This second assertion may look like an existential assertion but it isn't; the words "at most one" completely changes the meaning. The assertion"there is at most one $x \in T$ such that $P(x)$' can be restated as "Whenever x and y are members of T such that both $P(x)$ and $P(y)$ are true, we must have $x = y$" and this is actually a universal assertion:

For all $x \in T$ and $y \in T$, if $P(x)$ and $P(y)$ then $x = y$.

The assertion that $P(x)$ has a unique solution is abbreviated symbolically by adding an "!" after "$\exists$" as follows:

$$\exists! x \in T, P(x).$$

If the set T is understood from context, we write simply "$\exists! x, P(x)$".

Assertions with multiple variables. Assertions can have more than one variable. Each variable in the assertion is either free or dummy, and it's important to distinguish these. The assertion is definite if it has no free variables, otherwise it is indefinite. Here are some examples; in these sentences x and y are real number variables.

17. $x+y^2 \geq xy$. This has both x and y as free variables. It is an indefinite assertion depending on both x and y. We can abbreviate it symbolically by $B(x, y)$.

18. For all real numbers x, $x+y^2 \geq xy$. This has x as a bound variable and y as a free variable. It is an indefinite assertion depending on y (but not x) and is denoted symbolically by $\forall x \in \mathbb{R}, B(x, y)$. Since x is a bound variable we can reformulate the sentence without mentioning x: For every real number, the sum of the number and y^2 is at least the product of the number and y. We can't reformulate the sentence without y.

19. There exists a real number y such that $x+y^2 \geq xy$. This has y as a bound variable and x as a free variable. It is an indefinite assertion depending on x (but not y) and is denoted symbolically by $\exists y \in \mathbb{R}, B(x, y)$. Since y is a dummy variable we can reformulate the sentence without mentioning y: There is a real number with the property that the sum of x and the square of the chosen number is at least x times the chosen number.

$\forall\exists$ assertions and $\exists\forall$ assertions Building on the previous examples, we now discuss two extremely important types of assertions involving two variables, in which one variable has a universal quantifier, and the other has an existential quantifier

20. For all real numbers x, there exists a real number y such that $x + y^2 \geq xy$. This has both x and y as dummy variables, and no free variables, and so is a definite assertion. It is denoted symbolically by $\forall x \in \mathbb{R} \exists y \in \mathbb{R}, B(x, y)$. Since x and y are both dummy variables, we can reformulate this sentence without either x or y: For any real number, it is possible to choose a second number so that the first number plus the square of the second is at least the product of the two numbers.

21. There exists a real number y such that for all real numbers x, $x + y^2 \geq xy$. This has both x and y as dummy variables, and no free variables, and so is a definite assertion. It is denoted symbolically by $\exists y \in \mathbb{R} \forall x \in \mathbb{R}, B(x, y)$. Since x and y are both dummy variables, we can reformulate this sentence without either x or y: There is a real number, so that for any real number, For any real number, the first number plus the square of the second is at least the product of the two numbers.

Sentences 20 and 21 are examples of extremely common and important classes of assertions. Let's understand what each one means, and compare them.

Sentence 20 says: No matter what real number x is chosen, we can choose y so that $x + y^2 \geq xy$. Is this true? Let's try an example. If you choose $x = 10$ then it's easy to see that many values of y works, for example $y = 20$. If you choose $x = -5$ then again many choices of y work, for example $y = 5$ works. Notice that the value of y chosen may depend on x. It is easy to argue that for x negative we can take $y = 1$, since then the lefthand side is $x + y^2 = x + 1$ and the righthand side is $x(1) = x$, and so the lefthand side is larger than the righthand side. If instead $x \geq 0$ we just take $y = x$ and then the lefthand side is $x + x^2$ which is certainly at least the righthand side which is x^2.

Sentence 21 says something different. It says that it is possible to select a single y that works for all x. This turns out to be impossible. For example, if you try $y = 100$, it doesn't work for $x = 200$, and no matter which y you pick, there will be an x for which $B(x, y)$ is false.

A sentence whose form is like that of 20 is called a "for all-there exists" or $\forall\exists$ type of sentence, while a sentence whose form is like that of 21 is called a "there exists-for all" or $\exists\forall$ sentence.

We just saw that it is possible for the $\forall\exists$ sentence to be true while the $\exists\forall$ sentence is false. However, this can't happen the other way around. If a sentence $\exists y \forall x C(x, y)$ is true then the sentence $\forall x \exists y C(x, y)$ must also be true. To see this, assume that $\exists y \forall x C(x, y)$ is true. Then it is possible to pick y to be a fixed value, call it y_0 so that for every choice of x $C(x, y_0)$ holds. Now to verify that $\forall x \exists y C(x, y)$ we need to show that given any x we can choose a y that makes $C(x, y)$ true. So for any selected x we'll choose y to be y_0. We know $C(x, y_0)$ holds.

Solutions sets of assertions with more than one free variable We defined the solution set of predicate $P(x)$ where x has type T to be be $\{x \in T : P(x)\}$. We now make an analogous definition We can do the same with assertions that have more than one free variable. Suppose $A(x, y)$ is an assertion with free variables x of type T_1 and y of type T_2. A solution to $A(x, y)$ is now an *ordered pair* belonging to $T_1 \times T_2$. We define the solution set of $A(x, y)$ to be:

$\{(x, y) \in T_1 \times T_2 : A(x, y)\}$ to be the set of ordered pairs that, when substituted for x, y make $A(x, y)$ true.

We can also define other sets associated to $A(x, y)$. Sometimes we want to view one of the two variables as fixed. For example, if we hold y fixed T_2 we can ask for the set of $x \in T_1$ that make $A(x, y)$ true for that fixed y. We can denote this set by $\{x : A(x, y)\}$. In general this set depends on the fixed value of y, so it is customary to include y in the notation for the set. For example, we may define the set $S_y = \{x : A(x, y)\}$ for each $y \in T_2$.

We can think of $S = (S_y : y \in T_2)$ as an indexed family with index set T_2, where each S_y is a subset of T_1.

Alternatively, we can think of S as a function mapping T_2 to $\mathcal{P}(T_1)$.

Example 4.1. Suppose that $A(x, y)$ is the indefinite assertion $y \geq x^2$. Then the set $S(y) = \{x : x^2 \leq y\}$ is a set that varies with y. It is empty if $y < 0$ and is equal to the set of real numbers between $-\sqrt{y}$ and $\sqrt{y}$ if $y \geq 0$. Notice that S is a function that maps each $y \in \mathbb{R}$ to a subset of $\mathbb{R}$.

Building new assertions from old

There are various ways to modify and combine assertions to create new assertions. In the previous section we introduced *quantification*, which converts an assertion with a free variable into an existential or universal assertion. In this section we'll introduce some other important ways to change an assertion: *negation*, and *logical combination*.

The negation of an assertion Any assertion makes a specific claim that is either true or false (which for indefinite assertions may depend on the values of some variables.) Given an assertion A, the assertion "It is not the case that A" is called the *negation of the assertion* A, and is abbreviated $\neg A$.

For example:

22. The negation of the definite assertion "$59 \times 48 \geq 52^2$" is the sentence "It is not the case that $59 \times 48 \geq 52^2$" or "$59 \times 48 \geq 52^2$ is false"'

23. The negation of the definite assertion "Every even integer bigger than 2 can be expressed as the sum of two primes." is the sentence "It is not the case that every even integer bigger than 2 can be expressed as the sum of two primes"

24. The negation of the indefinite assertion "$x^3 + x \geq 5x^2$" is "It is not the case that $x^3 + x \geq 5x^2$".

The phrases "It is not the case that" and "is false" are *negating phrases* that convert a sentence to its negative. It is often useful to be able to reformulate the negated sentence without negating phrases. For example, Assertion 22 means the same as "$59 \times 48 < 52^2$". For Assertion 23 we can rewrite the negated sentence as "There is an even integer bigger than 2 that can't be expressed as the sum of two primes."

For an assertion A, $\neg A$ is true precisely when A is false and $\neg A$ is false when A is true.

Combining two assertions Suppose that A and B are assertions. There are four basic ways to combine A and B into a new assertion:

and ($\wedge$) The assertion "A and B" is true provided that both A and B are true. The symbolic abbreviation is $A \wedge B$.

or ($\vee$) The assertion "A or B" is true provided that at least one of A and B is true. The symbolic abbrevation is $A \vee B$.

implies, if-then ($\implies$) The assertion "A implies B", also written "if A then B" means that "if it is the case that A is true then B must also be true". This sentence is considered false if A is true and B is false, and is considered true otherwise. In particular, if A is false then $A \implies B$ is true, whether or not B is false. The symbolic abbreviation of "A implies B" is $A \implies B$.

if and only if ($\iff$) The assertion "A if and only if B", is true provided A and B are both true or A and B are both false. The symbolic abbreviation is $A \iff B$.

When we combine two assertions using one of these connectives, the truth value of the resulting assertion depends only on the connective used, and the truth value of the assertions being combined. The following table summarizes this:

$TV(A)$	$TV(B)$	$TV(A \wedge B)$	$TV(A \vee B)$	$TV(A \implies B)$	$TV(A \iff B)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

You can combine any two assertions by any of these methods. For example, from the two assertions: "7 is prime" and "13 is divisible by 4" we can build the following assertions:

25. 7 is prime and 13 is divisible by 4.

26. 7 is prime or 13 is divisible by 4.

27. if 7 is prime then 13 is divisible by 4.

28. if 13 is divisible by 4 then 7 is prime.

29. 7 is prime if and only if 13 is divisible by 4.

Since "7 is prime" is true and "13 is divisible by 4" is false, Assertion 25 is false, Assertion 26 is true, Assertion 27 is false and and Assertion 29 is false.

Remark 4.4. The mathematical use of "and", "or", "implies" and "if and only if" is similar to their use in everyday language, but there are some differences.

1. When someone says "John and I are going to the movie" then "and" does not connect two assertions, it connects two nouns "John" and "I". This sentence is a shortened way to write "John is going to the movie and I am going to the movie", which is the combination of two assertions with "and". A mathematician might write "5 and 7 are prime" or "5 or 6 is even" but it is important to realize that logically the first is shortened way to write "5 is prime and 7 is prime", and the second is a shortened way to write "5 is even or 6 is even".

2. In mathematics, "A or B" means that at least one of A and B is true, and possibly both are true. In everyday language, "A or B" might have a similar meaning, or it might mean that exactly one of A and B is true, for example, "I will have pizza for dinner or I will have sushi for dinner" usually means "I will have pizza for dinner or I will have sushi for dinner, but not both". When a mathematician who means "A or B but not both" must clearly say this or something like it, such as "Exactly one of A and B is true".

3. In a statement "if A then B", the assertion A is called the *assumption* and B is called the *conclusion* and the meaning is "if A is true then B is also true". The only way the sentence is considered false is if A is true and B is false. In everyday usage, when we say "if A then B" or "A implies B" we normally use it in a situation where A can be thought of as *causing* B. We might say, "if I miss the bus then I will be late for my appointment" Here the first part is the assertion "I miss the bus" while the second assertion is "I will be late for my appointment" and the first assertion is the cause of the second. In mathematics,"A implies B" is usually used when A and B are related, but the rules of logic don't require this, and "A implies B" even when A and B have no connection. Such a sentence is true unless A is true and B is false, in which case it is false.

Remark 4.5. **Combining indefinite assertions.** Indefinite assertions can be combined in the same way that definite assertions are. However, there are a few things to be aware of.

- Don't use the same letter as both a dummy variable and a free variable in a single sentence. Suppose we have the sentences "$x \leq 7$" and "For all x, $x^2 + 1 \geq x$. In the first sentence x is a free variable, while in the second it isi a dummy variable. If you combine these sentences, using "and" for instance, you get "$x \geq 7$ and for all x, $x^2 + 1 \geq x$. The use of x as both a free variable and a dummy variable in the same sentence is potentially confusing, and should not be done. Here's how to avoid it. Recall that in an assertion where x is a dummy variable we may replace all occurences of x by another letter. So replace all x's in the second sentence by a different letter, say z. The combined sentence will then be "$x \geq 7$" and for all z. $z^2 + 1 \geq z$.

- Consider the two sentences "If n is prime and $n > 2$ then n is odd." and "For all n, if n is prime and $n > 2$, then n is odd". From a mathematical standpoint these sentences are different. In the first sentence n is a a free variable. The second assertion is a definite assertion with dummy variable n. The second sentence is true. Since the first sentence is an indefinite assertion it does not have a truth value, but if we substitute a specific value for n it does have a truth value.

  Even though the first sentence is different from the second, it is common, even among mathematicians, to treat the first sentence as though it has "For all n" added to the beginning so that it means the same as the second sentence. This is an example of a violation of the "safety rules" of mathematical communication. It happens to be a violation that is not that dangerous (it's unlikely to cause confusion) but it is better for students in this course not to violate the safety rule and to treat these sentences as different.