14 Cardinality: Comparing the size of sets [15]

In previous chapters, we discussed finite and infinite sets, and introduced the notation $|A|$ to mean the size of the finite set A So far, we've relied on our intuitive idea of what a finite set is, and what we mean by the size of the set. In this section, we will take a more careful look at the idea of the "size" of a set. Our goals are:

- To provide a precise mathematical definition of the terms *finite set* and *infinite set.*
- To give a precise mathematical definition of the *size* of a finite set.
- To give a precise definition of what it means for one set to be larger than another, that applies to both finite and infinite sets.
- To consider the question: are all infinite sets the same size or are some infinite sets larger than others?

14.1 Using functions to compare sets

One of the major ideas underlying modern mathematics is that the relationship between two sets is best understood by studying functions from one to the other. This approach can be used to give a precise definition of what we mean when we say that one set is at least as large as another:

Definition 14.1. We say that set A *embeds in* B provided that there is a one-to-one function f with domain A and with $\textbf{Range}(f) \subseteq B$. In this case we say that f is an*embedding* of A into B.

An embedding of A to B associates each member of A to a different member of B, and so intuitively B is at least as large as A.

Proposition 14.1. *If $A \subseteq B$ then A embeds in B.*

Exercise 14.1. Prove Proposition 14.1.

Proposition 14.2. *The relationship "embeds" is transitive and reflexive.*

Exercise 14.2. Prove Proposition 14.2.

Exercise 14.3. Prove that the relation "embeds" is neither symmetric nor anti-symmetric.

The definition of "A embeds in B" involves one-to-one functions from A to B. It turns out we can also determine if A embeds in B by studying maps from B to A:

Theorem 14.3. *Suppose A and B are arbitrary nonempty sets. Then A embeds in B if and only if there is a function f with domain A such that $B \subseteq$ **Range**(f).*

The 2nd condition in the theorem looks something like the definition of "A embeds in B" but is actually quite different. The definition of "A embeds in B" says that there is a one-to-one function with domain A having range that is a subset of B. The condition in the theorem says there is a function (not necessarily a one-to-one function) with domain B (rather than A) such that A is a subset (not a superset) of the range.

Proof. Suppose A and B are arbitrary sets. We will prove (1) implies (2) and leave the proof of (2) implies (1) as an exercise.

Suppose f is a function with domain B such that $A \subseteq \mathbf{Range}(f)$. We show that there is a one-to-one function from A to B. Recall that for $a \in A$, $\mathbf{PreIm}_f(a)$ is the set $\{b \in B : f(b) = a\}$. Since $A \subseteq \mathbf{Range}(f)$, for every $a \in A$, $\mathbf{PreIm}_f(a)$ is nonempty. Define the function $g : A \longrightarrow B$ as follows: for each $a \in A$, choose $g(a)$ to be some member of $\mathbf{PreIm}_f(a)$. We claim that g is one-to-one. Suppose a_1, a_2 in A satisfies $g(a_1) = g(a_2)$. Let $c = g(a_1) = g(a_2)$. Then $c \in \mathbf{PreIm}_f(a_1)$ and $c \in \mathbf{PreIm}_f(a_2)$ and so $a_1 = f(c) = a_2$, so $a_1 = a_2$, as required to prove f is one-to-one. $\square$

Exercise 14.4. Complete the proof of Theorem 14.3 by proving that if A embeds in B then there is a function f with domain B such that $A \subseteq \mathbf{Range}(f)$.

Advanced remark 14.1. A key step in the proof that (1) was to construct $g : B \longrightarrow A$ by choosing $g(b)$ to be some member of $\mathbf{PreIm}_f(b)$. The claim that this function is well-defined requires a separate axiom is called the *axiom of choice* which says: If S is a function that maps each member of B to a nonempty subset of A, then there exists a function $g : B \longrightarrow A$ such that for all $b \in B$, $g(b) \in S(b)$. While this axiom seems obvious, it has some surprising consequences, and plays an important role in the foundations of mathematics.

14.2 Sets that are "equally large"

As mentioned the condition "A embeds in B" can be thought of as meaning "A is at most as large as B." Next, we want to use functions between A and B to define what it means for A and B to be "equally large". There are two natural ways to make such a definition.

Definition 14.2. Suppose A and B are sets.

bi-embeddable We say A and B are *bi-embeddable* if A embeds in B and B embeds in A.

equinumerous We say A is *equinumerous* to B if there is a bijection from A to B.

Both definitions intuitively represent the idea that two sets A and B are equally large. "A embeds in B" represents the idea that "B is at least as large as A" and "B embeds in A" represents the idea that "A is at least as large as B", so "A and B are bi-embeddable" should represent the idea "A and B are equally large".

Also, if there is a bijection from A to B, then this bijection matches the elements of A to the elements of B, so this also represents the idea that A and B are equally large.

So these two relationships both intuitively represent what it means for A and B to be equally large. The obvious question then is whether the "bi-embeddable" relation and the "equinumerous" relation are actually the same. It is easy to see that if A and B are equinumerous then they are bi-embeddable: If there is a bijection f from A to B then f is one-to-one function from A to B, and f is invertible, and f^{-1} is a one-to-one map from B to A.

The harder thing to see is that if A and B are bi-embeddable, then A and B are equinumerous. The following example illustrates why this is not obviously true:

Example 14.1. Let A be the the real interval $[0, 1]$ and B be the real interval $(0, 1]$. Since $B \subseteq A$, we have B embeds in A by Proposition 14.1. But also A embeds in B. Consider the function $f : [0, 1] \longrightarrow (0, 1]$ defined by $f(x) = (1 + x)/2$. In Exercise 14.5 you are asked to show that the range of this function is $[1/2, 1]$, and the function is one-to-one. Therefore $[0, 1]$ embeds in $(0, 1]$. So we have $[0, 1]$ and $(0, 1]$ are bi-embeddable. But are they equinumerous, that is, is there is a bijection between these two sets? The answer to this question is far from obvious.

Exercise 14.5. Prove that the function f in example 14.1 has range $[1/2, 1]$ and is one-to-one.

In spite of the difficulty illustrated by this example, it is nevertheless true that if A and B are bi-embeddable then there must be a bijection between them. This is a famous theorem that is named after the mathematicians who first discovered it.

Theorem 14.4. *(Schröder-Bernstein Theorem.) For any two sets A and B, if there is a one-to-one function from A to B and a one-to-one function from B to A then there is a bijection from A to B.*

The proof of Theorem 14.4 requires some work. In Section 14.6, we'll outline the proof, leaving parts of it as exercises for the reader.

This theorem implies what we hoped for: that the concepts "equinumerous" and "bi-embeddable" are equivalent: two sets are equinumerous if and only if they are bi-embeddable.

Theorem 14.4 is a very useful tool for showing that two sets are equinumerous. It is hard to find a bijection between the two sets in Example 14.1, but it's easy to show that each embeds in the other, so Theorem 14.4 tells us that this guarantees that there is a bijection between the two sets.

Let's look at some basic properties of the "equinumerous" relation.

Proposition 14.5. *The "equinumerous" relation is an equivalence relation on the set of sets.*

Exercise 14.6. Prove Proposition 14.5

Since "equinumerous" is an equivalence relation on the set of sets, we know that it partitions the set of all sets into equivalence classes.

Definition 14.3. The equivalence classes of the "equinumerous" relation are called *cardinals*. For a set A, the*cardinality* of A is the equivalence class containing A. (Two sets with the same cardinality are thought of as being "equally large".)

We can now define precisely what we mean when we say that one set is larger than another:

Definition 14.4. We say that set B is *larger than* set A provided that A embeds in B but A is not equinumerous to B.

Exercise 14.7. Prove that the "larger than" relation is a strict partial order on the set of sets.

Suppose we compare two sets A and B using embeddings. There are four possibilities:

- A embeds in B and B embeds in A, that is, A and B are equinumerous.
- A embeds in B and B does not embed in A, that is, B is larger than A.
- A does not embed in B and B embeds in A, that is, A is larger than B.
- A does not embed in B and B does not embed in A.

The following theorem tells us that the fourth possibility can't happen:

Theorem 14.6. *For any two sets A and B, A embeds in B or B embeds in A, and therefore A and B are either equinumerous, or A is larger than B, or B is larger than A.*

The proof of Theorem 14.6 will not be given here. While it is not that difficult, it involves some more advanced mathematical ideas that are not covered in this course.

14.3 Finite sets

Here is our precise definition of *finite set*, and *size of a finite set.*

Definition 14.5. Suppose A is a set.

1. We say that A is *finite* if there is an integer n such that A embeds in $\{1, \ldots, n\}$.
2. If A is finite we define the size of A, denoted $|A|$, to be the smallest integer n such that A embeds in $\{1, \ldots, n\}$.

Exercise 14.8. Apply the definition to find the size of the empty set.

Proposition 14.7. *For any set A and integer n, if A embeds in $\{1, \ldots, n\}$ then $|A| \leq n$.*

Exercise 14.9. Prove Proposition 14.7.

The following gives a useful way to compare the sizes of finite sets.

Proposition 14.8. *For any sets A and B, if B is finite and A embeds in B then $|A| \leq |B|$.*

Proof. Suppose A and B are arbitrary sets. Assume B is finite, and that A embeds in B. We must show $|A| \leq |B|$. We'll do this by showing that $|A|$ embeds in $\{1,\ldots,|B|\}$, and then apply Proposition 14.7. Since A embeds in B there is a one-to-one function we'll call f having domain A and satisfying $\mathbf{Range}(f) \subseteq B$. By definition of $|B|$, there is a one-to-one function we'll call g having domain B and satisfying $\mathbf{Range}(g) \subseteq \{1,\ldots,|B|\}$. Since $\mathbf{Range}(f) \subseteq \mathbf{Dom}(g)$, the composition $g \circ f$ is a well-defined function with domain A, and satisfies $\mathbf{Range}(g \circ f) \subseteq \mathbf{Range}(g) \subseteq \{1,\ldots,|B|\}$. Since both g and f are one-to-one, so is $g \circ f$. So $|A|$ embeds in$\{1,\ldots,|B|\}$ and by Proposition 14.7, $|A| \leq |B|$. ☐

For finite sets A and B, we now have two different mathematical definitions that represent the idea that A and B have the same size:

1. There is a bijection from A to B (so A and B are equinumerous).

2. $|A| = |B|$ where $|A|$ is as defined in Definition 14.5.

These two mathematical conditions are supposed to both represent the same idea, but it is not obvious from their definitions that they do. So we ask:

Question 14.10. For finite sets A and B,

1. If A and B are equinumerous does that imply $|A| = |B|$?

2. If $|A| = |B|$ does that imply that A and B are equinumerous?

The answer to both questions is yes. The following theorem answers the first question:

Theorem 14.9. *For any two finite sets A and B, if A and B are equinumerous then $|A| = |B|$.*

Exercise 14.11. Prove Theorem 14.9. (Hint: Use Proposition 14.8.)

To answer the second question, requires more work. Suppose A and B satisfy $|A| = |B|$ and suppose $s = |A| = |B|$. Our strategy will be to show that A and B are both equinumerous with $\{1,\ldots,s\}$, and since equinumerous is an equivalence relation, this will imply that A and B are equinumuerous.

To implement our proof strategy we must prove the following:

Proposition 14.10. *Suppose A is a nonempty finite set of size n. Then A is equinumerous with $\{1,\ldots,n\}$.*

Like many results in this chapter, this seems obvious, but the proof requires some work.

Proof. Suppose A is a nonempty finite set of size n. We must show that A is equinumerous with $\{1,\ldots,n\}$. By definition, $|A| = n$ implies that there is a one-to-one function we'll call f with domain A satisfying $\mathbf{Range}(f) \subseteq \{1,\ldots,n\}$. If we can prove that $\mathbf{Range}(f) = \{1,\ldots,n\}$ we'll be done, since then f is a bijection between A and $\{1,\ldots,n\}$ and so A and $\{1,\ldots,n\}$ are equinumerous. Suppose for contradiction that $\mathbf{Range}(f) \neq \{1,\ldots,n\}$. Since $\mathbf{Range}(f) \subseteq \{1,\ldots,n\}$, we must have that $\mathbf{Range}(f)$ is a proper subset of $\{1,\ldots,n\}$.

We'll show that there is a one-to-one function g with domain A and $\mathbf{Range}(g) \subseteq \{1, \ldots, n-1\}$. By Proposition 14.7 this implies $|A| \leq n-1$, contradicting $|A| = n$.

The instructions for constructing g depend on whether $n \in \mathbf{Range}(f)$.

If $n \notin \mathbf{Range}(f)$, then we define $g = f$. This is a one-to-one function with $\mathbf{Range}(g) \subseteq \{1, \ldots, n-1\}$.

If $n \in \mathbf{Range}(f)$, then there is a member of A we'll call b such that $f(b) = n$ and since f is one-to-one no other member of A is mapped to n. Since $\mathbf{Range}(f)$ is a proper subset of $\{1, \ldots, n\}$ there is a member of $\{1, \ldots, n\}$ we'll call j that is not in $\mathbf{Range}(f)$. Since $n \in \mathbf{Range}(f)$, we have $j < n$. Now define the function g on A by the rule: $g(b) = j$ and $g(x) = f(x)$ for $x \neq b$. (So g is obtained from f by having b map to j instead of n.) Note that b is the only member of A that g maps to j. To show g is one-to-one, suppose $x_1, x_2 \in A$ and $g(x_1) = g(x_2)$.

Case 1. Assume $x_1 = b$ or $x_2 = b$. Then $g(x_1) = g(x_2) = j$ and $x_1 = x_2 = b$ since b is the only member of A that g maps to j.

Case 2. Assume $x_1 \neq b$ and $x_2 \neq b$. Then $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$ so $f(x_1) = f(x_2)$ and since f is one-to-one, $x_1 = x_2$.

Also, $\mathbf{Range}(g) \subseteq \{1, \ldots, n-1\}$ since $\mathbf{Range}(g) \subseteq \mathbf{Range}(f) \cup \{j\}$ and by assumption $\mathbf{Range}(f) \subseteq \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n-1\}$. By Proposition 14.7, $|A| \leq n-1$. Since this contradicts $|A| = n$, we conclude that $\mathbf{Range}(f) = \{1, \ldots, n\}$ and therefore A is equinumerous with $\{1, \ldots, n\}$ as required. □

As noted above, Proposition 14.10 is all we needed to complete the proof of Theorem 14.9. We have therefore shown that for finite sets, the conditions $|A| = |B|$ and A is equinumerous with B are equivalent.

14.4 Mapping a set to a subset of itself

Suppose A is a set and B is a proper subset of A. It would seem that B is smaller than A, since A contains all of the members of B and additional members. However, when we compare sets using functions we find that in some cases B and A can be equally large!

Example 14.2. Consider the set $\mathbb{Z}$ and the subset $\mathbb{E}$ of even integers. Then $\mathbb{E}$ and $\mathbb{Z}$ are equinumerous. To see this consider the function $f : \mathbb{E} \longrightarrow \mathbb{Z}$ given by $f(n) = n/2$ for all $n \in \mathbb{E}$. This function is a bijection; you are asked to prove this in Exercise 14.12. Therefore $\mathbb{Z}$ and $\mathbb{E}$ are equinumerous.

Exercise 14.12. Prove that the function f defined in Example 14.2 is a bijection.

This example shows that our intuition is wrong: it is possible that a set can be embedded with one of its proper subset. But the intuition is correct for *finite sets*.

Theorem 14.11. *Suppose A is a finite set and B is a proper subset of A. Then:*

1. $|B| < |A|$.
2. *A can not be embedded into B.*

Exercise 14.13. Prove Theorem 14.11. (The results in the previous subsection may be helpful.)

14.5 Comparing infinite sets

We have seen that the set of integers is equinumerous with its proper subset of even integers, which shows that our intuition about finite sets does not always apply to inifinite sets. In this section we investigate the following question:

Question 14.14. Are any two infinite sets equinumerous, or are there infinite sets A and B where A is larger than B?

Let's consider another example: is the set $\mathbb{Z}$ of integers equinumerous with the set $\mathbb{Q}$ of rational numbers? Since $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Z}$ embeds in $\mathbb{Q}$. It certainly seems as though $\mathbb{Q}$ is much bigger than $\mathbb{Z}$, but we'll see that they are equinumerous. By Theorem 14.4 it is enough to show that $\mathbb{Q}$ embeds in $\mathbb{Z}$.

Theorem 14.12. *There is a one-to-one function from $\mathbb{Q}$ to $\mathbb{Z}$.*

Proof. We'll start by defining a one-to-one function $v : \mathbb{Q}_{>0} \longrightarrow \mathbb{Z}_{>0}$. Given a positive rational number r, we can write r as a ratio of positive integers in lowest terms $a(r)/b(r)$, so that $a(r)$ and $b(r)$ have no factors in common. We then define $v(r)$ to be $2^{a(r)}3^{b(r)}$. We claim that v is one-to-one. Suppose that r and q are rational numbers such that $v(r) = v(q)$. We must show $r = q$. Since $v(r) = v(q)$, $2^{a(r)}3^{b(r)} = 2^{a(q)}3^{b(q)}$. In Exercise 14.15 you are asked to complete the proof that v is one-to-one by showing that $r = q$ and in Exercise 14.16 you are asked to use the function v to construct a one-to-one function from $\mathbb{Q}$ to $\mathbb{Z}$. □

Exercise 14.15. Suppose r and q are positive rational numbers and that $r = a(r)/b(r)$ and $q = a(q)/b(q)$. If $2^{a(r)}3^{b(r)} = 2^{a(q)}3^{b(q)}$ then $r = q$.

Exercise 14.16. Use the function $v : \mathbb{Q}_{>0} \longrightarrow \mathbb{Z}_{>0}$ to construct a one-to-one function from $\mathbb{Q}$ to $\mathbb{Z}$.

So $\mathbb{Z}$ and $\mathbb{Q}$ are equinumerous. So let's enlarge $\mathbb{Q}$ even further and consider the set $\mathbb{R}$. Is $\mathbb{Z}$ equinumerous with $\mathbb{R}$? Now the answer is no:

Theorem 14.13. *$\mathbb{R}$ does not embed in $\mathbb{Z}$.*

To prove this it will be useful to bring a third set into the discussion, the set $\mathcal{P}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$. We will show two things:

1. $\mathcal{P}(\mathbb{Z})$ embeds in $\mathbb{R}$.
2. $\mathcal{P}(\mathbb{Z})$ does not embed into $\mathbb{Z}$.

Once we show these, it's easy to prove Theorem 14.13 . Suppose for contradiction that $f : \mathbb{R} \longrightarrow \mathbb{Z}$ is a one-to-one function. By (1), there is also a one-to-one function we'll call g from $\mathcal{P}(\mathbb{Z})$ to $\mathbb{R}$. Then $f \circ g$ is a one-to-one function from $\mathcal{P}(\mathbb{Z})$ to $\mathbb{Z}$, contradicting (2)

So it remains to prove (1) and (2).

To prove (1) we define a one-to-one function g from $\mathcal{P}(\mathbb{Z})$ to $\mathbb{R}$. Consider the real numbers between 0 and 1, written as infinite decimals. For each subset I of $\mathbb{Z}$, define $g(I)$ to be the number between 0 and 1 whose decimal representation has 1's in the positions indexed by I and 0 in the remaining positions. For example if I is the set of perfects squares $\{1, 4, 9, \ldots\}$ then $g(I)$ is the number $0.100100001000000100\ldots$. Every subset of $\mathbb{Z}$ maps to a different infinite decimal, so the function is one-to-one.

Remark 14.2. For this proof, we needed the fact infinite decimals correspond to real numbers, and two infinite decimals represent different real numbers. We haven't proved this yet, and in fact, it's not quite true. For example, the infinite decimals $.01\bar{0}$ and $.00\bar{9}$ (where the bar over a digit indicates that the digit is infinitely repeated) represent exactly the same number. However, it can be proved that the infinite decimals that arise from our function g do represent different numbers.

Next, we prove (2). This will follow from a more general fact:

Theorem 14.14. *For all sets S, there is no one-to-one function from $\mathcal{P}(S)$ to S.*

Proof. Suppose S is an arbitrary set. We must show that there is no one-to-one function from $\mathcal{P}(S)$ to S. Suppose $f : \mathcal{P}(S) \longrightarrow S$. We'll show that f is not one-to-one by finding two members of the domain A and B such that $f(A) = f(B)$. Say that a member s of S is *notable* if there is a subset T of S with $s \notin T$ such that $f(T) = s$. Let N be the set of notable elements of S and let $n = f(N)$. We will show that there is a set M that is different from N such that $f(M) = f(N) = n$, thus proving that f is not one-to-one.

First we claim that n is notable. For contradiction, assume $n \notin N$. But since $f(N) = n$ and $n \notin N$, we have that n is notable.

Since n is notable, we have $n \in N$. Also, by the definition of notable, there is a set we'll call M such that $n \notin M$ and $f(M) = n$. Then $f(M) = f(N)$ and since $n \in N$ and $n \notin M$ we have $M \neq N$. □

Theorem 14.14 implies that $\mathcal{P}(\mathbb{Z})$ does not embed in $\mathbb{Z}$, which is what we needed to prove (2) and complete the proof of Theorem 14.13

14.6 Proof of the Schröder-Bernstein theorem

In Section 14.1 we stated Theorem 14.4, which says that if A and B are bi-embeddable (meaning that there A embeds in B and B embeds in A) then they are equinumerous. Here we outline the proof of this theorem, leaving the various steps as exercises.

Let's set up the proof. Suppose that A and B are arbitrary sets and suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are one-to-one. We must show that there is a bijection $h : A \longrightarrow B$.

We start by making the additional assumption that $A \cap B = \emptyset$. After showing how to prove the theorem in this case, we'll show how to prove the general case.

Step 1. Define the function $h : A \cup B \longrightarrow A \cup B$ by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$. Define a *path* to be a sequence of elements $x_1, \ldots, x_k$ (where possibly $k = 1$) from $A \cup B$ such that for each $i \in \{1, \ldots, k-1\}$ we have $h(x_i) = x_{i+1}$. Let R be the relation on $A \cup B$ where xRy if there is a path that starts at x and ends at y or a path that starts at y and ends at x.

Lemma 14.15. *The relation R is an equivalence relation on $A \cup B$.*

Exercise 14.17. Prove Lemma 14.15.

We will use the notation $[x]_R$ to mean the equivalence class of x under this relation.

Step 2. We need to define a bijection from A to B. Our strategy will be to consider each R-equivalence class separately. For each equivalence class C, we'll define a bijection from $A \cap C$ to $B \cap C$. If we can do this for every equivalence class then we can combine these bijections together to get a bijection from A to B.

We consdier two special properties that may or may not hold in an R-equivalence class:

- C has property P_g if $C \cap A \subseteq \mathbf{Range}(g)$.
- C has property P_f if $C \cap B$ is a subset of $\mathbf{Range}(f)$.

For a R equivalence class C, let f_C be the restriction of f to C which means $f_C : A \cap C \longrightarrow B$ and for each $x \in A \cap C$. $f_C(x) = f(x)$.

Lemma 14.16. *Suppose C is an R-equivalence class.*

1. *f_C is one-to-one and $\boldsymbol{Range}(f_C) \subseteq B \cap C$.*
2. *If C satisfies P_f, then $\boldsymbol{Range}(f_C) = B \cap C$, and therefore f_C is a bijection from $A \cap C$ to $B \cap C$.*

Exercise 14.18. Prove Lemma 14.16

Exercise 14.19. Prove that if C does not satisfy P_f then $\mathbf{Range}(f_C) \neq B \cap C$ and therefore f_C is not a bijection from $A \cap C$ to $B \cap C$.

Exercise 14.19 shows that for equivalence classes C that don't satisfy P_f we can't use f_C as our bijection from $A \cap C$ to $B \cap C$. So we need a different method in this case.

Step 3. For an R equivalence class C, let g_C be the restriction of g to $B \cap C$ which means $g_C : B \cap C \longrightarrow A$ and for each $b \in B \cap C$. $g_C(b) = g(b)$.

Lemma 14.17. *Suppose C is an R-equivalence class.*

1. *g_C is one-to-one and $\boldsymbol{Range}(g_C) \subseteq A \cap C$.*
2. *If C satisfies P_g, then $\boldsymbol{Range}(g_C) = A \cap C$.*

Exercise 14.20. Prove Lemma 14.17. (Note: This is very similar to Exercise 14.18.)

We can therefore view g_C as a bijection from $B \cap C$ to $A \cap C$. Technically the target of g_C is all of A and we're changing the target, so it's a different function, which we'll call w_C. Since w_C is a bijection, it has an inverse w_C^{-1} which is a bijection from $A \cap C$ to $B \cap C$.

So we have shown:

Lemma 14.18.

- *If C satisfies P_f then f_C is a bijection from $A \cap C$ to $B \cap C$.*
- *If C satisfies P_g then w_C^{-1} is a bijection from $A \cap C$ to $B \cap C$.*

Step 5. Our strategy is to show that for every R-equivalence class C, we can construct a bijection from $A \cap C$ to $B \cap C$. Lemma 14.18 shows that we can do this for any equivalence class that satisfies P_f and for every equivalence class that satisfies P_g. We now claim that every equivalence class satisfies at least one of these properties.

Lemma 14.19. *For any R-equivalence class C, C satisfies P_f or C satisfies P_f.*

Exercise 14.21. Prove Lemma 14.19. (Hint: Use proof by contradiction.)

Step 6. We now have shown that for every equivalence class C we can define a bijection from $A \cap C$ to $B \cap C$. To finish the proof (for the case $A \cap B = \emptyset$, we need to combine these bijections into a single bijection from A to B.

Exercise 14.22. Finish the proof for the case $A \cap B = \emptyset$ by constructing a single bijection from A to B using Lemma 14.18 and Lemma 14.19. Your description of the bijection should have the following form:

$$b(x) = \begin{cases} \text{[expression involving } x\text{]} & \text{if [condition on } x\text{]} \\ \text{[expression involving } x\text{]} & \text{if [condition on } x\text{]} \end{cases}$$

Finally, we consider the general case (where $A \cap B$ need not be $\emptyset$). In this case define $A' = A \times \{0\}$ (so A' consists of all ordered pairs $(a, 0)$ where $a \in A$. Similarly define $B' = B \times \{1\}$. We complete the proof of the general case in the following exercise.

Exercise 14.23.

1. Show that if A and B are bi-embeddable then A' and B' are also bi-embeddable.
2. Since $A' \cap B' = \emptyset$, we can apply the result just proved to find a bijection from A' to B'. Use this bijection to find a bijection from A to B.