Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 9 [1]

1. (30 points) Say that a relation R on A is a TR relation if it is both transitive and reflexive. (TR relation is not a standard term.) We know that a TR relation that is symmetric is an equivalence relation, and a TR relation that is anti-symmetric is a partial order. The purpose of this problem is to give a general characterization of TR relations. We'll prove: A relation R on A is a TR relation if and only if there is a set B with a partial order relation Q, and a function $f : A \longrightarrow B$ such that for all $x, y \in A$, xRy if and only if $f(x)Qf(y)$.

   This is an "if and only if" theorem so there are two directions of the implication to prove: $\Rightarrow$ and $\Leftarrow$. The $\Leftarrow$ direction is easy and is the first part of the problem. The rest of the problem is to prove the $\Rightarrow$ directon.

   (Warm up problem, not to be handed in: Prove the theorem in the case that (a) R is an equivalence relation, and (b) R is a partial order relation.)

   (a) Suppose that Q is a partial order on B and that $f : A \longrightarrow B$. Define the relation R on A by xRy if $f(x)Qf(y)$. Prove that R is a TR relation.
   (b) For the rest of the problem, suppose R is an arbitrary TR relation on A. Define the relation W on A by xWy if and only if xRy and yRx. Prove that W is an equivalence relation.
   (c) Let $\mathcal{C}$ denote the set of equivalence classes of W. Define a relation P on the set $\mathcal{C}$ where CPD if there exists an $x \in C$ and a $y \in D$ such that xRy. Prove that in fact, for all $C, D \in \mathcal{C}$ if CPD then for all $x \in C$ and $y \in D$, xRy.
   (d) Prove that P is a partial order on $\mathcal{C}$.
   (e) Finish the proof of the $\Rightarrow$ direction.

2. Recall that if R is a relation on A, an R-chain is a list $(a_1, \ldots, a_k)$ such that for each $i \in \{2, \ldots, k\}$. $a_{i-1}Ra_i$. An R-chain is said to be *non-repeating* if all of the entries of the chain are different. An R-chain is said to be a *Hamilton path* (named after the mathematician William Rowan Hamilton) provided that it is non-repeating and every element of A appears on ths list. (Notice that this requires that A is finite.) Recall that a relation R on A is *full* if for all $x, y \in A$ if $x \neq y$ then xRy or yRx. Prove the following neat theorem: For any finite set A and full relation R on A, there is a Hamilton path in R. (A hint is given below.[2])

3. Define a relation R on $\mathbb{Z}_{>0}$ as follows: For $m, n \in \mathbb{Z}_{>0}$, we have mRn provided that there are odd numbers a and b so that $ma = nb$.

   (a) Prove that this relation is transitive, symmetric and reflexive, and is therefore an equivalence relation.
   (b) Let S be the set consisting of the smallest member from each R-equivalence class. Determine explicitly what the set S is (and prove that your answer is correct.)

4. (**20 points**) For a relation R on A, a subset X of A is R-independent provided that for all $x, y \in X$, if $x \neq y$ then $x \not R y$. Also, if A is finite, define $c(R)$ to be the length of the largest R-chain.

   The purpose of this problem is to prove the following theorem: Suppose A is a finite partially ordered set with partial order R. Then there exists a partition of A into exactly $c(R)$ sets, each of which is R-independent. (As a warm-up, test out the theorem on a few small partially ordered sets.)

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[2] Let $(a_1, \ldots, a_t)$ be an R-chain of longest length that is non-repeating. Assume for contradiction that not all members of A appear in this R-chain, and derive a contradiction.

(a) Define the function $h : A \longrightarrow \mathbb{Z}_{>0}$ as follows: $h(x)$ is the length of the longest chain that ends with x. Prove that for all $x \in A$, $1 \leq h(x) \leq c(R)$.

(b) Let $(a_1, \ldots, a_{c(R)})$ be an R-chain of length $c(R)$. Prove that for $i \in \{1, \ldots, c(R)\}$, $h(a_i) = i$. (Hint: Prove separately that $h(a_i) \geq i$ and $h(a_i) \leq i$.)

(c) Prove that for each $i \in \{1, \ldots, c(R)\}$, $h^{-1}(i)$ is an R-independent set. (Hint: Use proof by contradiction.)

(d) Finish the proof of the theorem.