Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 8 [1]

1. Suppose k is a positive integer and consider the relation $\equiv_k$ on $\mathbb{Z}$ defined for integers m and n by $m \equiv_k n$ provided that $k|(m-n)$. Prove that for any integers m_1, m_2, n_1, n_2 if $m_1 \equiv_k m_2$ and $n_1 \equiv_k n_2$ then (1) $m_1 + n_1 \equiv_k m_2 + n_2$ and (2) $m_1 n_1 \equiv_k m_2 n_2$.

2. Let Q be the relation on the set $\mathbb{R}$ consisting of all pairs $(x, y) \in \mathbb{R}^2$ satisfying $x + 1 < y$. Let P be the relation on the set $\mathbb{R}$ consisting of of all pairs $(x, y) \in \mathbb{R}^2$ satisfying $x < y + 1$. Prove or disprove:
   (a) Q is antisymmetric.
   (b) Q is transitive.
   (c) P is antisymmetric.
   (d) P is transitive.

3. (Recall that for real numbers x and y, $[x, y]$ is the set $\{z \in \mathbb{R} : x \leq z \leq y\}$.) Define $I[x, y]$ to be the set $[x, y] \cup [y, x] = [\min(x, y), \max(x, y)]$. Suppose that S is an arbitrary subset of $\mathbb{R}$. Define the relation R_S on S where xR_Sy provided that $I[x, y] \subseteq S$. Prove that R_S is an equivalence relation. (The proof of transitivity requires some care.)

4. Suppose Q is a relation on A. A Q-*chain* is a list $(a_1, a_2, \ldots, a_k)$ of elements of A such that for each i between 1 and $k-1$, $a_i Q a_{i+1}$. For elements a and b of A we say that b is Q-*reachable* from a provided that there is a Q-chain that starts at a and ends at b. Define Q^* to be the relation on A consisting of those pairs (a, b) such that b is Q-reachable from a.
   (a) Prove: For any relation Q on A, Q^* is transitive.
   (b) Prove or disprove: For any relation Q on A, if Q is symmetric then so is Q^*.
   (c) Prove or disprove: For any relation Q on A if Q is antisymmetric then so is Q^*.

5. (Note: In this problem we will be defining a relation on the set $A \times A$. The elements of $A \times A$ are ordered pairs, so a pair of the relation we define is an *ordered pairs each of whose coordinates is an ordered pairs.* This can be confusing, so be sure to read and think about the problem carefully.) Suppose A is a set and R a partial order relation on A. Define a new relation Q on the set $A \times A$ as follows: For $a_1, a_2, b_1, b_2 \in A$ we have $(a_1, a_2)Q(b_1, b_2)$ provided that ($a_1 \neq b_1$ and $a_1 R b_1$) or ($a_1 = b_1$ and $a_2 R b_2$).
   (a) Prove that Q is a partial order on the set $A \times A$.
   (b) Prove that if R is a total order (which means that for all $a, b \in A$ we have aRb or bRa) then so is Q.

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