Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 7 [1]

1. Prove or disprove: For all functions $f : S \longrightarrow T$ and $g : T \longrightarrow U$ if $g \circ f$ is onto then g is onto.

2. Prove or disprove: For all functions $f : S \longrightarrow T$ and $g : T \longrightarrow U$ if $g \circ f$ is onto then f is onto.

3. Prove: For any functions $f : S \longrightarrow T$ and $g : T \longrightarrow S$ if $|S| \geq 2$ and g is the unique left inverse for f then g is also a right inverse for f. (Hint: Prove the contrapositive.)

4. Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. We say that a set $Z \subseteq X$ is a *lower bound* for $\mathcal{S}$ provided that $Z \subseteq S$ for all $S \in \mathcal{S}$. Prove that for any set X and any collection $\mathcal{S}$ of subsets of X, there is a unique set T with the following two properties:

   (a) T is a lower bound for $\mathcal{S}$.

   (b) For any set Z that is a lower bound for $\mathcal{S}$ we have $Z \subseteq T$.

   (As usual, when you are given an abstract problem like this, try a couple of simple examples to help you understand what it means.)

The last (many part) problem is intended to give you practice working with the concept of *relation* and some basic definitions that are at the beginning of section 10. You can try them on your own, or wait until I introduce the material in class on Monday.

Here's a brief summary of the relevant definitions. A *relation* R is a list consisting of three sets $(\mathbf{Source}(R), \mathbf{Target}(R), \mathbf{pairs}(R))$, where $\mathbf{pairs}(R) \subseteq \mathbf{Source}(R) \times \mathbf{Target}(R)$. In words, $\mathbf{Source}(R)$ and $\mathbf{Target}(R)$ can be any two sets (possibly the same) and $\mathbf{pairs}(R)$ is a set of ordered pairs with first coordinate in $\mathbf{Source}(R)$ and second coordinate in $\mathbf{Target}(R)$. If $S = \mathbf{Source}(R)$ and $T = \mathbf{Target}(R)$ we say that R is a *relation from S to T*.

For a relation R, and $s \in \mathbf{Source}(R)$ and $t \in \mathbf{Target}(R)$ we use the notation sRt to mean that $(s,t) \in \mathbf{pairs}(R)$, and $s \not R t$ to mean $(s,t) \notin \mathbf{pairs}(R)$.

A relation Q with the same source and target A is called a *relation on A*. We say that Q is:

reflexive provided that for all $a \in A$, we have aQa.

antireflexive provided that for all $a \in A$ we have $a \not Q a$.

transitive provided that for all $a, b, c \in A$ if aQb and bQc then aQc.

symmetric provided that for all $a, b \in A$ if aQb then bQa.

anti-symmetric provided that for all $a, b \in A$ if $a \neq b$ and aQb then $b \not Q a$.

The purpose of the following problem is to have you work carefully through the above definitions on some examples. They are not difficult, but you need to carefully apply the definitions to figure them out.

5. (20 point problem) For each of the following relations, determine which of the properties *reflexive*, *anti-reflexive*, *transitive*, *symmetric*, and *anti-symmetric* it satisfies. If the property is not satisfied, give a counterexample; if it's satisfied provide a proof.

   (a) Let $\mathcal{S}$ be a collection of non-empty subsets of a set X and let R be the relation on $\mathcal{S}$ with $\mathbf{pairs}(R)$ consisting of all pairs $(S,T) \in \mathcal{S} \times \mathcal{S}$ satisfying $S \cap T = \emptyset$.

   (b) Let $\mathcal{S}$ be a collection of non-empty subsets of a set X and let R be the relation on $\mathcal{S}$ with $\mathbf{pairs}(R)$ consisting of all pairs $(S,T) \in \mathcal{S} \times \mathcal{S}$ satisfying $S \cap T \neq \emptyset$.

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(c) Let R be a relation on $\mathbb{Z}$ defined so that for $m, n \in \mathbb{Z}$, $(m, n) \in R$ provided there are nonzero integers r and s so that $mr^2 = ns^2$.

(d) Let $S = \mathbb{R} \times \mathbb{R}$ and let R be the relation defined as follows for $(x_1, x_2) \in S$ and $(y_1, y_2) \in S$, we have $(x_1, x_2)R(y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 > y_2$. (Careful, this one may be confusing because the set S consists of ordered pairs, so **pairs**(R) is a set of ordered pairs, and for each ordered pair in **pairs**(R) each of its coordinates is an ordered pair.)