Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 6 [1]

1. (20 point problem.) Suppose $f : S \longrightarrow T$. Recall that for $X \subseteq S$, the *image* of X under f, denoted $im_f(X)$ is the set $\{f(x) : x \in X\}$ and for $Y \subseteq T$, the preimage of Y under f, denoted $preim_f(Y)$ is the set $\{x \in S : f(x) \in Y\}$.

   For each of the following assertions determine whether it's true or false. If it's true prove it. If it's false, disprove it.

   (a) For any two subsets X_1 and X_2 of S, $im_f(X_1 \cup X_2) = im_f(X_1) \cup im_f(X_2)$.
   (b) For any two subsets X_1 and X_2 of S, $im_f(X_1 \cap X_2) = im_f(X_1) \cap im_f(X_2)$.
   (c) For any two subsets Y_1 and Y_2 of T, $preim_f(Y_1 \cap Y_2) = preim_f(Y_1) \cup preim_f(Y_2)$.
   (d) For any subset X of S, $preim_f(im_f(X)) \supseteq X$.
   (e) For any subset X of S, $preim_f(im_f(X)) = X$.
   (f) For any subset Y of T, $im_f(preim_f(Y)) \supseteq Y$.
   (g) For any subset Y of T, $im_f(preim_f(Y)) = Y$.

   (Hint: As usual, to get some intuition about each statement, construct some examples of a function with a finite domain and target and see what's true in the examples.)

2. A rational number is defined to be a real number r that can be written as a ratio of two integers. Formally, this means that r is rational provided that there exist integers a and b such that $r = a/b$. The set of rational numbers is denoted $\mathbb{Q}$ and $\mathbb{Q}_{\geq 0}$ is the set of nonnegative rational numbers.

   (a) Consider the function $f : \mathbb{Z}_{>0} \longrightarrow \mathbb{R}$ defined by the following rule: on input $n \in \mathbb{Z}_{>0}$ let a be the largest integer such that 2^a is a divisor of n. Let $b = \lceil \frac{n}{2^{a+1}} \rceil$, where, for any real number x, $\lceil x \rceil$ is defined to be the least integer that is greater than or equal to x. Then $f(n) = \frac{a}{b}$. Prove that the range of f is $\mathbb{Q}_{\geq 0}$.
   (b) Intuitively, the set of rational numbers seems much bigger than the set of integers. What does the result of the previous part say about this intuition. (This question is not a fomal math problem: I'm asking you to think about what the previous result means and comment on it.)
   (c) Suppose someone makes the following definition: Let $g : \mathbb{Q} \longrightarrow \mathbb{Z}$ be the function defined by $g(a/b) = a + b$. Is this a proper function definition? Why or why not?

3. A set $\mathcal{S}$ of sets is said to be *intersecting* if for any two members A and B of $\mathcal{S}$, we have $A \cap B \neq \emptyset$. Prove that for any nonempty set U and for any intersecting collection $\mathcal{S}$ of subsets of U and for any $X \subseteq U$ at least one of the two collections $\mathcal{S} \cup \{X\}$ and $\mathcal{S} \cup \{U \backslash X\}$ is intersecting.

continued...

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4. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *even* if for every $x \in \mathbb{R}$, $f(-x) = f(x)$ and is said to be *odd* if for every $x \in \mathbb{R}$, $f(-x) = -f(x)$. (Not to hand in: Find examples of an even function and an odd function that are given by polynomials, and find examples that are not polynomials.) Also for functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$, then the functions $f + g : \mathbb{R} \longrightarrow \mathbb{R}$ and $f - g : \mathbb{R} \longrightarrow \mathbb{R}$ are defined by the rules $(f + g)(x) = f(x) + g(x)$ and $(f - g)(x) = f(x) - g(x)$. Prove that every function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is the sum of an odd function and an even function. This footnote[2]provides a hint.

[2]Hint: Consider the two functions r and s given by the rules $r(x) = h(x) + h(-x)$ and $s(x) = h(x) - h(-x)$.