Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 5 [1]

1. Recall the following definition: For any two sets A and B, the *difference set* $A \backslash B$ is the set consisting of those objects that are members of A but not members of B. Also $A \triangle B$ is equal to $A \backslash B \cup B \backslash A$.

   (a) Prove or disprove: For all sets A, B, C, if $A \backslash C = B \backslash C$ then $A = B$.

   (b) Prove or disprove: For all sets A, B, C, if $A \triangle C = B \triangle C$ then $A = B$.

2. Recall that if S is a set, a *partition* of S is a set $\mathcal{P}$ of nonempty subsets of S satisfying the following two conditions: (i) for each $s \in S$, there is an $M \in \mathcal{P}$ such that $s \in M$, and (ii) For each pair $M_1, M_2 \in \mathcal{P}$ such that $M_1 \neq M_2$, we have $M_1 \cap M_2 = \emptyset$.

   Now suppose X is a nonempty set. For $\mathcal{H}, \mathcal{G} \subseteq \mathcal{P}(X)$ define $\mathcal{H} \sqcap \mathcal{G}$ to be the set $\{A \cap B : A \in \mathcal{H}, B \in \mathcal{G}, A \cap B \neq \emptyset\}$.

   (a) Construct an example of two different partitions $\mathcal{H}$ and $\mathcal{G}$ of $\{1, 2, 3, 4, 5, 6\}$ each having two parts, and construct $\mathcal{H} \sqcap \mathcal{G}$.

   (b) Prove: for any two partitions $\mathcal{H}$ and $\mathcal{G}$ of X, $\mathcal{H} \sqcap \mathcal{G}$ is also a partition. (In your proof, carefully verify the definition that $\mathcal{H} \sqcap \mathcal{G}$ satisfies the definition of partition.)

3. The following theorem appears as part 1 of Corollary 7.23 in the notes. For any two functions$f : B \longrightarrow C$ and $g : A \longrightarrow B$ if f is one-to-one and g is one-to-one then $f \circ g$ is one-to-one. A proof is given in the notes, using the equivalence of one-to-one and left-invertibility. Give an alternative proof that works directly from the definition of one-to-one.

4. .

   (a) Prove: For any sets A, B, C and D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   (b) Show that if we replace $\cap$ in all three places by $\cup$ in the previous assertion, then it is false.

   (c) Prove: For any sets A,B,C, and D, $(A \times B) \cup (C \times D) \supseteq (A \cap C) \times (B \cup D)$.

5. Let X be a set and let $\mathcal{H}$ be a set of of subsets of X. For elements s, t of X we define the notation $s \approx_{\mathcal{H}} t$, to mean that there is a member of $\mathcal{H}$ that has both s and t as a member.

   (a) Prove: For any set S and for any partition $\mathcal{P}$ of S and for any $s, t, u \in S$, if $s \approx_{\mathcal{P}} t$ and $t \approx_{\mathcal{P}} u$ then $s \approx_{\mathcal{P}} u$.

   (b) Suppose that in the previous statement, we replace the phrase "for any partition $\mathcal{P}$ of S" by "for any set $\mathcal{P}$ of subsets of S". Prove that the resulting statement is false.

   (c) Now go back to your proof of part a. Try to modify your proof for part a, so that it proves the statement in part b. Since the statement in part b is false, this should be impossible, which means that somewhere in your proof there must be a point where it is crucial that $\mathcal{P}$ is a partition and not just a set of subsets of S. Find where this occurs in your proof. (Note: If you can't do this, then there's something wrong with your proof and you should fix it!)

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6. We begin with some definitions. We say that A *is a neighbor of* B if $A \Delta B$ consists of exactly one element.

   A list of sets is a *neighborly list of sets* if each set is a neighbor of the set following it in the list, and the last set is a neighbor of the first set.

   Here is an interesting theorem: For all positive integers n, there is a list consisting of subsets of $\{1, \ldots, n\}$ such that (1) every subset of $\{1, \ldots, n\}$ appears *precisely once* on the list and (2) the list is neighborly.

   Prove the special cases of the theorem with $n = 1$, $n = 2$, $n = 3$ and $n = 4$. (We'll prove the theorem for all n later, but if you feel ambitious you can try it now.)

7. TO DO, BUT NOT HAND IN: Exercises from Section 7: 7.3, 7.4, 7.6, 7.7, 7.8.