Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 4 [1]

1. For each of the following assertions, identify the free variables and the bound variables.
   (a) (Variables: integers n, m, r,) For every positive integer n the set $\{m \in \mathbb{Z} : m^2 - r \text{ is divisible by } n\}$ is nonempty.
   (b) (Variables: real numbers x, y, ε, and subset $S \subseteq \mathbb{R}$.) x is not a member of S and for all real numbers $\varepsilon > 0$, there exists a member y of S such that $|x - y| \leq \varepsilon$.
   (c) (Variables: functions f, g and h and real number x) There is a function g and a functon h such that for every real number x, $f(x) = g(x) + h(x)$ and $g(x) = g(-x)$ and $h(-x) = -h(x)$.

2. (20 point problem.) Below several pairs of assertions are given. For each pair do the following:
   - Identify the atomic assertions common to each pair of assertions and assign a variable to each of these assertions. (Note: if two atomic assertions have opposite meaning such as "$y \leq 3$" and " "$y > 3$" you should represent one as an assertion $S(y)$ and the other as $\neg S(y)$ rather than give them two different letters.)
   - Find logical expressions for each sentence in terms of the variables.
   - Determine whether the first can be logically deduced from the second, and whether the second can be logically deduced from the first. Explain your answers.

   Note 1. You are *not asked* to determine the truth or falsity of any of these sentences.

   Pair 1 In the two sentences: a positive integer n is *composite* if there are two integers different from 1 whose product is n.
   (a) (n is prime or $n + 2$ is prime) implies that $n^2 + 2$ is prime or $n^2 - 2$ is prime
   (b) $n^2 + 2$ is composite and $n^2 - 2$ is composite implies n is composite and $n + 2$ is composite.

   Pair 2 (a) For all real numbers x, there is a real number y such that $y^2 + y + 10x = 0$, or $x \leq 9$ and there is a real number z such that $z^2 + 2z + 15x = 0$.
   (b) For all real numbers x, $x \leq 9$ or there is both a real number y such that $y^2+y+10x = 0$ and a real number z such that $z^2 + 2z + 15x = 0$.

   Pair 3 (a) $f(x) > y$ and $g(y) > x$ implies $f(g(y)) > y$ and $g(f(x)) > x$.
   (b) $f(g(y)) \leq y$ implies $f(x) \leq y$, and $g(f(x)) \leq x$ implies $g(y) \leq x$.

   Pair 4 In this pair of assertions, S, T, V, and W are all sets.
   (a) $S \subseteq T$ if and only if $S \subseteq V$, or $S \subseteq T$ if and only if $W \subseteq T$
   (b) $S \subseteq T$ if and only if ($S \subseteq V$ or $W \subseteq T$).

3. (a) Give an example of a collection of 4 sets such that any two distinct sets in the collection intersect in exactly one element, and no element belongs to more than 2 sets.
   (b) Generalize the previous example: For each positive integer k give an example of a collection of k sets such that any two distinct members of the collection intersect in exactly one element and no element belongs to more than 2 sets.

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4. Let x and y be real variables, and let $C(x, y)$ be an indefinite assertion involving x and y. Consider the two scenarios:

   Scenario 1 Input variable: x. Assumption: For every $y \in \mathbb{R}$, $C(x, y)$ is true.

   Scenario 2 Input variable y. Assumption: For every $x \in \mathbb{R}$, $C(x, y)$ is false.

   Observe that in the first scenario y is a bound variable, while in the second x is a bound variable. Consider the set S_1 of feasible instances to scenario 1 and S_2 of feasible instances to scenario 2.

   Can S_1 and S_2 both be nonempty? Explain.