Intro to Mathematical Reasoning (Math 300)–Section H2
Assignment 2 [1]

Constructing examples of mathematical objects with desired properties. Building examples is one of the basic ways to build your understanding of mathematical objects. Most students have not had much experience doing this. The following problems are intended to give you some practice with it. If you haven't done this much before, it can be a bit tricky. Usually this problem asks you to build a certain mathematical object of a particular type that meets some requirements. There are no formulas for doing this. Building examples is a combination of trial and error,reasoning, and ingenuity. Given such a problem, build an example that meets some of the requirements. If it meets all of the requirements, great! If not, try to modify your example until it meets all requirements. If this doesn't work, start again with a completely different example. See me in office hours for hints.

Include with your example a brief explanation why your example does what its supposed to.

1. Recall: if a and b are integers then we say that *a is a divisor of b* provided that b/a is an integer. Gien a set J of integers, a common divisor of J is a number c that is a divisor of each of the number in J.
   (a) Construct an example of a set of three integers that has no common divisor greater than 1, but any two integers in the set hae a common divisor greater than 1.
   (b) Generalize the previous example. Explain how, given the positive integer $k \geq 3$ you can construct a set of k integers that has no common divisor greater than 1, but any subset of size $k-1$ does have a common divisor greater than 1.
2. Given an example of a positive integer k that satisfies $2^k > k^{1000} + 1000000$. (Don't forget to give an explanation why your choice works.)
3. Give an example of a pair of functions $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ and $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ such that f and g are continuous, $\lim_{x \longrightarrow \infty} f(x) = \infty$, and there is an infinite sequence of numbers $x_1 < x_2 < \cdots < x_n < \cdots$ such that $f(x_j) > g(x_j)$ for j even and $f(x_j) < g(x_j)$ for j odd. (Hint: try to sketch graphs of a possible choice of f and g first.)
4. Give an example of an infinite sequence $(A_j : j \in \mathbb{Z}_{>0})$ that satisfies the three requirements: (1) Each A_j is a nonempty subset of the set $[-100, 100]$. (Recall that $[-100, 100]$ is the set of real numbers that are at least -100 and at most 100.) (2) For each $j \geq 1, A_{j+1} \subseteq A_j$, (3) There is no number that belongs to all of the sets.
5. If $(a_1, a_2, \ldots, a_n)$ is a list, a *sublist* of length k is obtained by selecting indices $j_1 < j_2 < \cdots < j_k$ between 1 and n and constructing the list $(a_{j_1}, a_{j_2}, \ldots, a_{j_k})$. (Intuitively, a sublist removes some entries, keeping the remaining entries in the original.) A sublist is *increasing* if each successive term is greater than the previous and *decreasing* if each successive term is less than the previous. For example $(1, 7, 4, 6, 8, 3, 9)$ has $(1, 4, 6, 9)$ as an increasing sublist.
   (a) Give an example of a list of length 6 with all entries distinct, that has no increasing sublist of length 4 and no decreasing sublist of length 3.
   (b) Generalize the previous example. Given two arbitrary positive integers s and t, construct an example of a list with st distinct entries that has no increasing sublist of length $s+1$ and no decreasing sublist of length $t+1$.

[1] Version: 9/9/2015

6. If S is a set, a *partition* of S is a set $\mathcal{P}$ of nomempty subsets of S satisfying the following two conditions: (i) for each $s \in S$, there is an $M \in \mathcal{P}$ such that $s \in M$, and (ii) For each pair $M_1, M_2 \in \mathcal{P}$ such that $M_1 \neq M_2$, we have $M_1 \cap M_2 = \emptyset$. The members of the partition are called the *parts* of the partition.

   (a) Find all possible partitions of $\{1, 2, 3, 4\}$.

   (b) Given an example of a set of subsets of $\{1, 2, 3, 4, 5\}$ that satisfies (i) but not (ii).

   (c) Give an example of a set of subsets of $\{1, 2, 3, 4, 5\}$ that satisfies (ii) but not (i).

   (d) Give an example of a partition of the set $\{1\}$.

   (e) Is it possible to have a partition of set $\{\}$. Why or why not?

7. (a) Give an example of a partition of $\mathbb{Z}_{>0}$ that has two parts, both of which have infinite size.

   (b) For each positive integer k, describe an example of a partition of $\mathbb{Z}_{>0}$ into k parts, all of which have infinite size.

   (c) Give an example of a partition of $\mathbb{Z}_{>0}$ that has an infinite number of parts, each of which has infinite size.

8. (a) If S is a finite set, a permutation of S is a function from S to itself, whose range is all of S. Give an example of a permutation of $\{1, 2, 3, 4, 5\}$.

   (b) Recall that if f and g are permutations of S then $f \circ g$ is the function from S to S given by the rule $f \circ g(s) = f(g(s))$ for all $s \in S$. The identity permutation is the permutation that maps every element to itself. Give an example of two different permutations f and g of $\{1, 2, 3, 4, 5\}$ such that neither is the identity permutation, and $f \circ g = g \circ f$.

   (c) Give an example of two different permutations f and g of $\{1, 2, 3, 4, 5\}$ such that neither is the identity permutation and such that $f \circ g = g \circ f$ and $f \circ f = g \circ g$.

9. Recall that for a three-dimensional box, with side length a,b and c, the volume is abc, the surface area is $2(ab + bc + ac)$ and the total length of all edges) is $4(a + b + c)$. (If you don't remember this, draw yourself a picture and check it.) Provide an example of two boxes, Box 1 and Box 2, where Box 1 has larger volume than Box 2, Box 1 has larger total edge length than Box 2, but Box 1 has smaller surface area than Box 2.

10. (This problem is a different type of problem from the others.) Let S denote the set $\{2, 10, 11, 18, 19, 27\}$. For each of the following statements, determine whether the statement is true or false. Give a careful explanation for each answer. (Note: "careful" does not mean "long".)

    Recall that when we say that an integer a *is a multiple of* b or b is a divisor of a provided that a/b is an integer.

    (a) For all elements x belonging to S there is a y belonging to S such that $x + y$ is a multiple of 5.

    (b) There exists an element x belonging to S such that for all elements y belonging to S, $x + y$ is a multiple of 5.

    (c) For all x belonging to S there is a y belonging to S such that $x + y$ is a multiple of 7.