Intro to Mathematical Reasoning (Math 300:H2)–Honors
Assignment 12 [1]

1. Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are lists of real numbers and let $x_0 = y_0 = x_{n+1} = y_{n+1} = 0$. Prove:
   (a) $\sum_{i=1}^n (x_{i+1} - x_{i-1})y_i = \sum_{i=1}^n (y_{i-1} - y_{i+1})x_i$.
   (b) $\sum_{i=1}^n (x_{i+1} - 2x_i + x_{i-1})y_i = \sum_{i=0}^n (y_{i+1} - y_i)(x_i - x_{i+1})$.

   Be sure to explain carefully any changes of indices you do in the summations.

2. Prove that for any positive real numbers x and y, $\sqrt{2(x+y)} \geq \sqrt{x} + \sqrt{y}$. (Note: This inequality, and those that follow should be proved without using calculus.)

3. Prove that for any real numbers x and y and real number λ between 0 and 1, that
$$(\lambda x + (1-\lambda)y)^2 \leq (\lambda x^2 + (1-\lambda)y^2).$$
   (If you're having trouble, try the case $\lambda = 1/2$ first.)

4. Recall that for a sequence $x_1, \ldots, x_n$ of positive real numbers,
$$\begin{aligned} AM(x_1, \ldots, x_n) &= \frac{1}{n}\sum_{i=1}^n x_i \\ GM(x_1, \ldots, x_n) &= (\Pi_{i=1}^n x_i)^{1/n} \\ HM(x_1, \ldots, x_n) &= \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}\right)^{-1}. \end{aligned}$$
   Here are two theorems:

   Theorem A For any list $x_1, \ldots, x_n$ of positive real numbers $AM(x_1, \ldots, x_n) \geq GM(x_1, \ldots, x_n)$
   Theorem B For any list $x_1, \ldots, x_n$ of positive real numbers $GM(x_1, \ldots, x_n) \geq HM(x_1, \ldots, x_n)$.

   We'll prove Theorem A later in class. In this problem, assume Theorem A, and use it to prove Theorem B.

5. Suppose that M is an n by n symmetric matrix and that $x_1, \ldots, x_n$ is a list (vector) of real numbers. Let I denote the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$. Prove: If every row sum of M is 0, and all of the entries off the diagonal are nonpositive, then for any list (vector) $x_1, \ldots, x_n$
$$\sum_{(i,j)\in I} x_i M_{i,j} x_j \geq 0.$$
   (Hints: (1) Prove that under the given assumptions,
$$\sum_{(i,j)\in I} x_i M_{i,j} x_j = -\frac{1}{2}\sum_{(i,j)\in I} (x_i - x_j)^2 M_{i,j}.$$
   (2) If you're having trouble, try the case $n = 2$ first.)

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6. Consider the set $\mathbb{Z}_m$ consisting of the equivalence classes under the relation $\equiv_m$, together with the operations $+_m$ and $\times_m$. (There are m equivalence classes, denoted $[0], [1], \ldots, [m-1]$. It is often customary to drop the brackets around "$[i]$" and write simply "i".) As mentioned in class, $\mathbb{Z}_m$ satisfies the field axioms if and only if m is prime. The key difference between the prime and non-prime case is the existence of multiplicative inverses:

   (a) Prove that if m is prime, then for every $i \in \{1, \ldots, m-1\}$ there is a $j \in \{1, \ldots, m-1\}$ such that $i \times j \equiv_m 1$.

   (b) Prove that if m is not prime, then there is an $i \in \{1, \ldots, m-1\}$ for which there is no $j \in \{1, \ldots, m-1\}$ for which $i \times j \equiv_m 1$.

7. Let X be a totally ordered set. We say that X has the *least upper bound property* property if every subset of X that is bounded above has a least upper bound. We say that X has the *greatest lower bound property* if every subset of X that is bounded below has a greatest lower bound. We say that X satisfies the *no gaps property* if for every pair of subsets A and B of X, if for all $a \in A$ and $b \in B$ we have $a \leq b$ then there exists a $c \in X$ so that for all $a \in A$, $a \leq c$ and for all $b \in B$, $c \leq b$. (Note: the terminology "no gaps property" is not standard.)

   Prove that the following three conditions on X are equivalent: X has the least upper bound property, X has the greatest lower bound property, and X has the no gaps property.