Intro to Mathematical Reasoning (Math 300:H2)–Honors
Assignment 11 [1]

1. Recall that a number is *perfect* if the sum of its proper divisors is equal to the number itself. Prove the following: if n is a positive integer such that $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

2. Recall that $\gcd(m, n)$ is the greatest common divisor of m and n and $\mathbf{lcm}(m, n)$ is the least common multiple of m and n. Prove that for any two positive integers m and n, $\gcd(m, n) \times \mathbf{lcm}(m, n) = m \times n$.

3. Suppose b is an integer greater than 1. Prove that for every positive integer n, there is a unique list $(n_0, n_1, \ldots, n_k)$ with each $n_i \in \{0, \ldots, b-1\}$ such that $n = \sum_{i=0}^{k} n_i b^i$. (So you are being asked to prove that every positive integer has a unique base b representation. Refer to Section 12.4 of the notes.)

4. Here is an algorithm that takes as input two positive integers m and n and outputs an integer. (Recall that for numbers a, b, $\max(a, b)$ is the maximum of a and b and $\min(a, b)$ is the minimum of a and b.)

   1 Let $g = \max(m, n)$.

   2 Let $s = \min(m, n)$.

   3 If s is a divisor of g then output s and stop.

   4 Otherwise, let r be the remainder when g is divided by s.

   5 Change the value of g to the value of s.

   6 Change the value of s to the value of r.

   7 Go to line 3.

   Prove that this algorithm outputs the greatest common divisor of m and n. (Hint: use induction on the maximum of m and n.)

5. Let $(b_1, m_1), \ldots, (b_k, m_k)$ be a sequence of pairs of integers where $m_i \geq 1$ for all i and consider the system of congruences with variable n: for each $i \in \{1, \ldots, k\}$, $n \equiv_{m_k} b_k$. Suppose n_0 is a solution to the system. Prove that for all integers t, t is a solution to the system if and only if $t - n_0$ is a multiple of $\mathbf{lcm}(m_1, \ldots, m_k)$.

6. If A and B are sets, we say that A embeds in B if there is a one-to-one function from A to B. We say A *strictly embeds* in B if A embeds in B and B does not embed in A. Prove that the relation *strictly embeds* is a strict partial order on the set of sets.

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