Intro to Mathematical Reasoning (Math 300)–Honors
Assignment 10 [1]

1. Use PMI to prove the following: For any positive integer n, and real number r, and real number b, if $r \neq 1$ then $\sum_{i=0}^{n-1} br^i = \frac{b(1-r^n)}{1-r}$. (Hint: hold b and r fixed, and do induction on n).

2. Prove that for all positive integers n, $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$.

3. (**20 points**) : A kth order recurrence equation is a *linear homogeneous constant coefficient (LHCC) recurrence* if there are real numbers $a_1, a_2, \ldots, a_k$, called the *coefficients of the recurrence*, with $a_k \neq 0$, and a constant $T \geq k$ such that for every $n \geq T$, $s_n = a_1 s_{n-1} + a_2 s_{n-2} + \cdots + a_k s_{n-k}$. The number T tells us the smallest n for which the equation is valid. For $n < T$, the values of the sequence must be specified in some other way.

   The *characteristic polynomial* of the aboveLHCC recurrence is the polynomial The *characteristic polynomial* of the recurrence R is the polynomial $p(x) = x^k - \sum_{i=1}^{k} a_i x^{k-i}$.

   In the following problem, $a_1, \ldots, a_k$ are fixed real numbers with $a_k \neq 0$, and R is the LHCC $s_n = a_1 s_{n-1} + a_2 s_{n-2} + \cdots + a_k s_{n-k}$ for $n \geq k$. (This LHCC is not fully specified since we don't have the initial conditions for $s_0, \ldots, s_{k-1}$.

   (a) Prove that if r is a root of the characteristic polynomial then the sequence given by $s_n = r^n$ is a solution to the LHCC.

   (b) Suppose that $r_1, \ldots, r_j$ are all roots of the characteristic polynomial. Prove: For any choice of constants $c_1, \ldots, c_j$, the sequence whose nth term is $\sum_{i=1}^{j} c_i r_i^n$ satisfies recurrence R. (If we are given initial considitions, we can try to choose the c_i to make the initial conditions true).

   (c) Consider the Fibonacci sequence given by $f_0 = f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Use the above method to find an explicit formula for f_n as a function of n alone.

4. In this problem we will prove two familiar rules about exponentiation. If r is a real number, r^n can be defined to be the nth term in the recurrence given by $s_0 = 1$ and for $n \geq 1$, $s_n = r \times s_{n-1}$.

   (a) Use the recurrence and PMI to prove that for all nonnegative integers m and n, $r^{m+n} = r^m \times r^n$. (Hint: Formulate the statement to be proved as "For all nonnegative integers m, it is the case that for all nonnegative integers n, $r^{m+n} = r^m \times r^n$. Prove this for each fixed m using induction on n.)

   (b) Prove: For all nonnegative integers m and n, $(r^m)^n = r^{mn}$. (Hint: Again hold m fixed an prove the result by induction on n.)

5. A pair of sets A and B are said to be *neighbors* if their symmetric difference $A \triangle B$ has size exactly one. A list of sets $A_1, \ldots, A_t$ is *neighborly* if for each $i \in \{1, \ldots, t-1\}$, A_i and A_{i+1} are neighbors. Prove that for all $n \geq 1$, we can form a neighborly list of the subsets of $\{1, \ldots, n\}$ in such a way that every subset of $\{1, \ldots, n\}$ appears in the list exactly once.

6. Let $a_1, \ldots, a_k$ be a list of integers. Define the function f that maps a list of k integers to an integer by the rule $f(x_1, \ldots, x_k) = a_1 x_1 + \cdots + a_k x_k$. Let R be the range of the function f. Prove:

   (a) For all $m, n \in R$ we have $m + n \in R$.

   (b) For all $n \in R$. and for all integers c we have $cn \in R$.

[1] Version:11/14/16