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Explicit Hilbert spaces for certain unipotent representations II

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0. Introduction

To each real semisimple Jordan algebra, the Tits-Koecher-Kantor theory associates a distinguished parabolic subgroup $P = LN$ of a semisimple Lie group G. The groups P which arise in this manner are precisely those for which N is abelian, and P is conjugate to its opposite $\overline{P}$.

Each non-open L-orbit $\mathcal{O}$ on N^* admits an L-equivariant measure $d\mu$ which is unique up to scalar multiple. By Mackey theory, we obtain a natural irreducible unitary representation $\pi_{\mathcal{O}}$ of P, acting on the Hilbert space

$$\mathcal{H}_{\mathcal{O}} = L^2(\mathcal{O}, d\mu).$$

In this context, we wish to consider two problems:

1. Extend $\pi_{\mathcal{O}}$ to a unitary representation of G.
2. Decompose the tensor products $\pi_{\mathcal{O}} \otimes \pi_{\mathcal{O}'} \otimes \pi_{\mathcal{O}''} \otimes \cdots$

If the Jordan algebra is Euclidean (i.e. formally real) then G/P is the Shilov boundary of a symmetric tube domain. In this case, the first problem was solved in [S1], [S2], where it was shown that $\pi_{\mathcal{O}}$ extends to a unitary representation of a suitable covering group of G. The second problem was solved in [DS], where we established a correspondence between the unitary representations of G occurring in the tensor product, and those of a "dual" group G' acting on a certain reductive homogeneous space. This correspondence agrees with the θ-correspondence in various classical cases, and also gives a duality between E_7 and real forms of the Cayley projective plane.

In this paper we start to consider these two problems for *non-Euclidean* Jordan algebras. The algebraic groundwork has already been accomplished in [S3], however the analytical considerations are much more subtle, and here we only treat the case of the representation $\pi_1 = \pi_{\mathcal{O}_1}$ corresponding to the *minimal* L-orbit $\mathcal{O}_1$.

It turns out that in order for the first problem to have a positive solution, one has to exclude certain Jordan algebras of rank 2. This is related to the Howe-Vogan result on the non-existence of minimal representations for certain orthogonal groups.

To each of the remaining Jordan algebras we attach a restricted root system Σ of rank n, where n is the rank of the Jordan algebra. The root multiplicities, d and e, of Σ play a decisive role in our considerations. For the reader's convenience, we include a list of the corresponding groups G and the multiplicities in the appendix.

For these groups, we show that π_1 extends to a spherical unitary representation of G, and that the spherical vector is closely related to the *one* variable Bessel K-function $K_\tau(z)$, where

$$\tau = \frac{d - e - 1}{2}.$$

The function $K_\tau(z)$ can be characterized, up to a multiple, as the unique solution of the modified Bessel equation

$$\psi'' + z^{-1}\psi' - \left(1 + \frac{\tau^2}{z^2}\right)\psi = 0$$

that decays (exponentially) as $z \to \infty$; and, to us, one of the most delightful aspects of the present consideration is the unexpected and uniform manner in which this classical differential equation emerges from the structure theory of G.

More precisely, we establish the following result:

We identify N with its Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ via the exponential map. We also fix an invariant bilinear form on $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, which is a certain multiple of the Killing form, normalized as in Definition 1.1 below. We use this form to identify N^* with $\overline{\mathfrak{n}} = \mathrm{Lie}(\overline{N})$. For y in $\overline{\mathfrak{n}}$, $\langle -\theta y, y\rangle$ is positive, and we define

$$|y| = \sqrt{\langle -\theta y, y\rangle}.$$

Theorem 0.1. *π_1 extends to a unitary representation of G with spherical vector $|y|^{-\tau}K_\tau(|y|)$.*

Since π_1 is spherical, its Langlands parameter is its infinitesimal character, and this can be determined via the (degenerate) principal series imbedding described in Section 2 below. It is then straightforward to verify that π_1 is the minimal representation of G, with annihilator equal to the Joseph ideal. (For $G = GL(n)$, the minimal representation is not unique.)

Thus our construction should be compared to other realizations of the minimal representations in [Br], [T], [H] etc. Although our construction is for a more restrictive class of groups, it does offer two advantages over the other constructions. The first advantage is that our construction works for a larger class of representations, and the second advantage is that it is well-suited for tensor product computations.

Both of these features will be explored in detail in a subsequent paper. In the present paper, we consider k-fold tensor powers of π_1, where k is strictly smaller than n (rank of Σ), and show that the decomposition can be understood in terms of certain reductive homogeneous spaces

$$G_k/H_k\,,\ 1 < k < n.$$

These spaces are defined in Section 3, and are listed in the appendix.

We consider also the corresponding Plancherel decomposition:

$$L^2(G_k/H_k) = \int_{\widehat{G}_k}^{\oplus} m(\kappa)\kappa\, d\mu(\kappa)\,,$$

where $d\mu$ is the Plancherel measure, and $m(\kappa)$ is the multiplicity function. Then we have

Theorem 0.2. *For $1 < k < n$, there is a correspondence θ_k between $\widehat{G}_k$ and $\widehat{G}$, such that*

$$\pi_1^{\otimes k} = \int_{\widehat{G}_k}^{\oplus} m(\kappa)\theta_k\,(\kappa)\, d\mu(\kappa).$$

1. Preliminaries

The results of this section are all well-known. Details and proofs may be found in [S1], [KS] and in the references therein (in particular, [BK] and [Lo]).

1.1. Root multiplicities. Let G be a real simple Lie group and let K be a maximal compact subgroup corresponding to a Cartan involution θ. We shall denote the Lie algebras of G, K etc by $\mathfrak{g}$, $\mathfrak{k}$ etc. Their complexifications will be denoted by lowercase fraktur letters with subscript $_{\mathbb{C}}$. Fix θ, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition.

The parabolic subgroups $P = LN$ obtained by the Tits-Kantor-Koecher construction are those such that N is abelian, and P is G-conjugate to its opposite parabolic

$$\overline{P} = \theta(P) = L\overline{N}.$$

In this case N has a natural structure of a real Jordan algebra, which is unique up to a choice of the identity element.

In (Lie-)algebraic terms, this means that P is a maximal parabolic subgroup corresponding to a simple (restricted) root α which has coefficient 1 in the highest root, and which is mapped to $-\alpha$ under the long element of the Weyl group.

In this situation, $M := K \cap L$ is a symmetric subgroup of K (this is *equivalent* to the abelianness of N), and we fix a maximal toral subalgebra $\mathfrak{t}$ in the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$.

The roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ form a restricted root system of type C_n, where $n = \dim_{\mathbb{R}} \mathfrak{t}$ is the (real) rank of N as a Jordan algebra (this result is essentially due to C. Moore). We fix a basis $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of $\mathfrak{t}^*$ such that

$$\Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \{\pm(\gamma_i \pm \gamma_j)/2, \pm\gamma_j\}.$$

The restricted root system $\Sigma = \Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ is of type A_{n-1}, C_n or D_n, and the first of these cases arises precisely when N is a Euclidean Jordan algebra. This case was studied in [S1], therefore we restrict our attention to the last two cases.

The root multiplicities in Σ play a key role in our considerations. If Σ is C_n, there are two multiplicities, corresponding to the short and long roots, which we denote by d and e, respectively. If Σ is D_n, and $n \neq 2$, then there is a single multiplicity, which we denote by d, so that D_n may be regarded as a special case of C_n, with $e = 0$.

The root system D_2 is reducible (being isomorphic to $A_1 \times A_1$) and *a priori* there are two root multiplicities. In what follows, we explicitly exclude the case when these multiplicities are different. This means that we exclude from consideration the groups

$$G = O(p, q), N = \mathbb{R}^{p-1,q-1} (p \neq q);$$

indeed, our main results are false for these groups. When the two multiplicities *coincide*, we once again denote the common multiplicity by d.

The multiplicity of the short roots $\pm(\gamma_i \pm \gamma_j)/2$ in $\sum(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is equal to $2d$, and the multiplicity of the long roots $\pm\gamma_i$ is $e + 1$.

In the appendix we include a table listing the groups under consideration, as well as the values of d and e for each of these groups.

1.2. Cayley transform. We briefly review the notion of the Cayley transform. Let C be the following element (of order 8) in $SL_2(\mathbb{C})$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The Cayley transform of $\mathfrak{sl}_2(\mathbb{C})$ is the automorphism (of order 4) given by

$$c = \operatorname{Ad} C.$$

It transforms the "usual" basis of $\mathfrak{sl}_2(\mathbb{C})$

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

to the basis

$$X = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix},\ Y = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix},\ H = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $X = c(x) = C^{-1}xC$, etc. In turn, c can be expressed as

$$c = \exp\ \mathrm{ad}\ \frac{\pi i}{4}(x+y) = \exp\ \mathrm{ad}\ \frac{\pi i}{4}(X+Y).$$

The key property of the Cayley transform is that it takes the compact torus (spanned by iH) to the split torus spanned by h (cf. [KW]).

We turn now to the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. By the Cartan-Helgason theorem the root spaces $\mathfrak{p}_{\gamma_j}$ are one-dimensional, and so by the Jacobson-Morozov theorem we get holomorphic homomorphisms

$$\Phi_j : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{g}_{\mathbb{C}},\ j = 1, ..., n$$

such that $X_j = \Phi_j(X)$ spans $\mathfrak{p}_{\gamma_j}$.

We fix such maps Φ_j, and denote the images of x, X, y, Y, h, H by x_j, X_j, etc. Since the roots γ_j are strongly orthogonal, the triples $\{X_j, Y_j, H_j\}$ commute with each other, and the Cayley transform of $\mathfrak{g}$ is defined to be the automorphism

$$c = \exp\ \mathrm{ad}\ \frac{\pi i}{4}\left(\sum X_j + \sum Y_j\right) = \prod \exp\ \mathrm{ad}\ \frac{\pi i}{4}(X_j + Y_j).$$

Thus we obtain an $\mathbb{R}$-split toral subalgebra $\mathfrak{a}$ defined by

$$\mathfrak{a} = c^{-1}(i\mathfrak{t}) = \mathbb{R}h_1 \oplus \cdots \oplus \mathbb{R}h_n.$$

The roots of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ are

$$\Sigma(\mathfrak{a}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \left\{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j\right\} \text{ where } \varepsilon_i = \frac{1}{2}\gamma_i \circ c.$$

The short roots have multiplicity $2d$ and the long roots have multiplicity $e+1$.

In fact $\mathfrak{a} \subset \mathfrak{l}$, and we have

$$\Sigma(\mathfrak{a}, \mathfrak{l}) = \left\{\pm(\varepsilon_i - \varepsilon_j)\right\},\ \Sigma(\mathfrak{a}, \mathfrak{n}) = \left\{\varepsilon_i + \varepsilon_j, 2\varepsilon_j\right\},$$

$$\Sigma(\mathfrak{a}, \overline{\mathfrak{n}}) = \left\{-\varepsilon_i - \varepsilon_j, -2\varepsilon_j\right\}$$

Definition 1.1. *The invariant form $\langle .,. \rangle$ on $\mathfrak{g}$ is normalized by requiring*

$$\langle x_1, y_1 \rangle = 1.$$

For $y \in \overline{\mathfrak{n}}$, we set $|y| \stackrel{\text{def}}{=} \sqrt{-\langle y, \theta y\rangle}$, as in Introduction.

1.3. Orbits and measures. We now describe the orbits of L in $\overline{\mathfrak{n}} \simeq N^*$. For $k = 1, ..., n-1$, define

$$\mathcal{O}_k = L \cdot (y_1 + y_2 + \ldots + y_k).$$

Then these, together with the trivial orbit $\mathcal{O}_0$, comprise the totality of the singular (i.e., non-open) L-orbits in $\overline{\mathfrak{n}}$.

We define $\nu \in \mathfrak{a}^*$ as

$$\nu = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n.$$

Then ν extends to a character of $\mathfrak{l}$, and we will write e^ν for the corresponding (spherical) character of L.

Lemma 1.2. *The orbit $\mathcal{O}_1$ carries a natural $L-$equivariant measure $d\mu_1$, which transforms by the character $e^{2d\nu}$, that is*

$$\int_{\mathcal{O}_1} g(l \cdot y) d\mu_1(y) = e^{2d\nu}(l) \int_{\mathcal{O}_1} g(y) d\mu_1(y).$$

Proof. Let S_1 be the stabilizer of y_1 in L. It suffices to show that the modular function of S_1 is the restriction, from L to S_1, of the character $e^{2d\nu}$. Passing to the Lie algebra $\mathfrak{s}_1$, we need to show that

$$\operatorname{tr} \operatorname{ad}_{\mathfrak{s}_1} = 2d\nu|_{\mathfrak{s}_1}.$$

To see this, we remark that $\mathfrak{s}_1$ has codimension 1 inside a maximal parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{l}$, corresponding to the stabilizer of the line through y_1. The space of characters of $\mathfrak{q}$ is two-dimensional, and it follows that the space of characters of $\mathfrak{s}_1$ is one-dimensional. Hence any character of $\mathfrak{s}_1$ is determined by its restriction to $\mathfrak{a} \cap \mathfrak{s}_1 = \operatorname{Ker} \varepsilon_1$. The restriction of ν to $\mathfrak{s}_1$ is nontrivial, hence

$$\operatorname{tr} \operatorname{ad}_{\mathfrak{s}_1} = k\nu$$

for some constant k.

Obviously, $\operatorname{tr} \operatorname{ad}_{\mathfrak{l}} = 0$, and the only root spaces missing from $\mathfrak{s}_1$ are the root spaces $\mathfrak{l}_{\varepsilon_1 - \varepsilon_j}$, $j \geq 2$ (each of these root spaces has dimension $2d$). Hence, for $a \in \mathfrak{a}$

$$\operatorname{tr} \operatorname{ad}_{\mathfrak{s}_1}(a) = -2d \sum_{j=2}^{n} (\varepsilon_1 - \varepsilon_j)(a),$$

and restricting this to $\operatorname{Ker} \varepsilon_1$, we obtain $2d\nu\,|_{\mathfrak{a} \cap \mathfrak{s}_1}$. □

Example. Consider $G = O_{2n,2n}$ realized as the group of all $2n \times 2n$ real matrices preserving the split symmetric form $\begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix}$. Then $P = LN = GL_{2n}(\mathbb{R}) \ltimes \mathrm{Skew}_{2n}(\mathbb{R})$. More precisely,

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{t^{-1}} \end{pmatrix} : A \in GL_{2n}(\mathbb{R}) \right\}$$

and

$$N = \left\{ \begin{pmatrix} I_{2n} & 0 \\ B & I_{2n} \end{pmatrix} : B + B^t = 0 \right\}.$$

Then

$$\mathfrak{a} = \{\mathrm{diag}(a_1, a_1, a_2, a_2, \dots, a_n, a_n, \\ -a_1, -a_1, -a_2, -a_2, \dots, -a_n, -a_n),\ a_i \in \mathbb{R}\}$$

is the toral subalgebra of $\mathfrak{g}$ (and $\mathfrak{l}$) described in the preceding subsection. We can take

$$y_1 = \begin{pmatrix} 0_{2n} & B_1 \\ 0 & 0_{2n} \end{pmatrix}, \text{ where } B_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{s}_1$ of the stabilizer $S_1 = \mathrm{Stab}_L\, y_1$ can be written as

$$\mathfrak{s}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} : A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in \mathfrak{sl}_2, A_{22} \in \mathfrak{gl}_{2n-2} \right\}$$

It is a codimension 1 subalgebra of the parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{gl}_{2n}$, where $\mathfrak{q} = (\mathfrak{gl}_2 + \mathfrak{gl}_{2n-2}) + \mathbb{R}^{2,2n-2}$.

Remark. In this example $\nu = \frac{1}{2}\,\mathrm{tr}$, $d = 2$ and $e^{2d\nu} = (\det)^2$.

2. Minimal representation of G

If χ is a character of $\mathfrak{l}$, we write π_χ for the (unnormalized) induced representation $\mathrm{Ind}_P^G(\chi)$. These representations were studied in [S3] in the "compact" picture, by algebraic methods. Among the results established there was the existence of a finite number of "small", unitarizable, spherical subrepresentations, which occur for the following values of χ

$$\chi_j = e^{-jd\nu},\ \ j = 1, \dots, n-1.$$

In this paper we use analytical methods, and work primarily with the "non-compact" picture, which is the realization of π_χ on $C^\infty(N)$, via the Gelfand-Naimark decomposition

$$G \approx N\overline{P}.$$

In fact, using the exponential map we can identify $\mathfrak{n}$ and N, and realize π_χ on $C^\infty(\mathfrak{n})$.

We will show that the unitarizable subrepresentation of π_{χ_1} admits a natural realization on the Hilbert space $L^2(\mathcal{O}_1, d\mu)$. Since there is no obvious action of G on this space, we have to proceed in an indirect fashion. The key is an explicit realization of the spherical vector σ_{χ_1}.

2.1. The Bessel function. We let d, e be the root multiplicities of $\Sigma(\mathfrak{t}, \mathfrak{k})$ as in previous section, and define

$$\tau_G = \tau = (d - e - 1)/2$$

as in the introduction.

Let K_τ be the K-Bessel function on $(0, \infty)$ satisfying

$$z^2 K_\tau'' + z K_\tau' - (z^2 + \tau^2) K_\tau = 0. \tag{1}$$

Put $\phi_\tau(z) = \dfrac{K_\tau(\sqrt{z})}{\left(\sqrt{z}\right)^\tau}$, then ϕ_τ satisfies the differential equation

$$D\phi_\tau = 0, \text{ where } D\phi = 4z\phi'' + 4(\tau + 1)\phi' - \phi. \tag{2}$$

We lift ϕ_τ to an M-invariant function g_τ on $\mathcal{O}_1$, by defining

$$g_\tau(y) = \phi_\tau(-\langle y, \theta y\rangle) = \frac{K_\tau(|y|)}{|y|^\tau}. \tag{3}$$

Remark. If $d = e$ (as is the case for $G = Sp_{2n}(\mathbb{C})$ or $Sp_{n,n}$), then $\tau = -\frac{1}{2}$ and

$$g_\tau(y) = |y|^{1/2} K_{-1/2}(|y|) = |y|^{1/2} \frac{\exp(-|y|)}{|y|^{1/2}} = e^{-|y|}.$$

If $d = e + 1$ (this is true for $GL_{2n}(\mathbf{k})$, $\mathbf{k} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), then

$$g_\tau(y) = K_0(|y|).$$

Proposition 2.1. (1) *g_τ is a (square-integrable) function in $L^2(\mathcal{O}_1, d\mu_1)$.*

(2) *The measure $g_\tau d\mu_1$ defines a tempered distribution on $\overline{\mathfrak{n}}$.*

Proof. (1) We define

$$\mathcal{O}' \stackrel{\text{def}}{=} \{y' \in \mathcal{O}_1 : |y'| = 1\}.$$

Then $\mathcal{O}'$ is compact; the map

$$\mathcal{O}' \times (0, \infty) \ni (y', w) \longmapsto wy' \in \mathcal{O}_1$$

is a diffeomorphism, and the measure $d\mu_1$ can be decomposed as a product

$$d\mu_1(wy') = d\mu'(y') d\mu''(w)$$

We now determine the explicit form of $d\mu''(w)$.

Define $h = \sum_{i=1}^{n} h_i$, then $(\operatorname{ad} h)y = -2y$ for any $y \in \overline{\mathfrak{n}}$. We take $y \in \mathcal{O}_1$, $z > 0$, $a = \ln z$ and calculate

$$\begin{aligned} d\mu_1(zy) &= d\mu_1\Big(\exp\Big(-a\frac{h}{2}\Big)\cdot y\Big) = e^{-2d\nu(-\frac{ah}{2})}d\mu_1(y) \\ &= e^{dna}d\mu_1(y) = z^{dn}d\mu_1(y). \end{aligned}$$

Therefore, for $z > 0$

$$d\mu_1(zy) = z^{dn}d\mu_1(y) \tag{4}$$

and it follows that $d\mu''(zw) = z^{dn}d\mu''(w)$, and so, up to a scalar multiple,

$$d\mu''(w) = w^{dn-1}dw,$$

where dw is the Lebesgue measure.

We can now calculate

$$\begin{aligned} \int_{\mathcal{O}_1} |g_\tau(y)|^2\, d\mu_1(y) &= \int_0^\infty \int_{\mathcal{O}'} \frac{K_\tau(w)^2}{w^{2\tau}} d\mu'(y')w^{dn-1}dw \\ &= c\int_0^\infty \frac{K_\tau(w)^2}{w^{2\tau}} w^{dn-1}dw, \end{aligned} \tag{5}$$

where $c = \mu'(\mathcal{O}')$ is a positive constant. The function $K_\tau(w)$ has a pole of order τ at 0 (or, in case of $\tau = 0$, a logarithmic singularity at 0), and it decays exponentially as $w \to \infty$ [W, 3.71.15]. Hence $w^{-2\tau}K_\tau(w)^2$ has a pole of order

$$4\tau = 2(d-e-1) \leq 2d-2 < dn-1$$

(recall that we require $n \geq 2$). Thus the integrand in (5) is non-singular and decays exponentially as $w \to \infty$. Therefore, the integral (5) converges and $g_\tau(y) \in L^2(\mathcal{O}_1, d\mu_1)$.

(2) From the calculation in (1), we see that $g_\tau(y) \in L^1_{\mathrm{loc}}(\mathcal{O}_1, d\mu_1)$ and has exponential decay at ∞ (i.e., as $|y| \to \infty$). This implies the result. □

We can now define the Fourier transform of g_τ,

$$\Phi = \widehat{g_\tau d\mu_1}$$

as a (tempered) distribution on $\mathfrak{n}$. The key result is the following

Proposition 2.2. *Φ is a multiple of the spherical vector σ_{χ_1}.*

The proof of this proposition will be given over the next two subsections.

2.2. Characterization of spherical vectors. For $\phi : \mathfrak{n} \to \mathfrak{n}$, let $\xi(\phi)$ denote the corresponding vector field:

$$\xi(\phi) f(x) = \frac{d}{dt} f(x + t\phi(x))\bigg|_{t=0} \text{ for } f : \mathfrak{n} \to \mathbb{C}.$$

Then we have the following formulas for the action of π_χ on $C^\infty(\mathfrak{n})$:

- for $x_0 \in \mathfrak{n}$, $\pi_\chi(x_0) = \xi(x_0)$,
- for $h_0 \in \mathfrak{l}$, $\pi_\chi(h_0) = \chi(h_0) - \xi([h_0, x])$,
- for $y_0 \in \overline{\mathfrak{n}}$, $\pi_\chi(y_0) = \chi[x, y_0] - \frac{1}{2}\xi([h, x])$, where $h = [x, y_0]$.

We need a Lie algebra characterization of σ_χ:

Lemma 2.3. *The space of $\pi_\chi(\mathfrak{k})$-invariant distributions on $\mathfrak{n}$ is 1–dimensional (and spanned by σ_χ).*

Proof. It is well known (and easy to prove) that the only distributions on $\mathbb{R}^n$, which are annihilated by $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$ are the constants. More generally, we can replace $\mathbb{R}^n$ by a manifold, and $\left\{\frac{\partial}{\partial x_i}\right\}$ by any set of vector fields which span the tangent space at each point of the manifold.

For $\chi = 0$, the formulas above show that $\pi_0(\mathfrak{g})$ acts by vector fields on $C^\infty(\mathfrak{n})$. Moreover, using the decomposition $G = K\overline{P}$, we see that $\pi_0(\mathfrak{k})$ is a spanning family of vector fields. Thus the result follows in this case.

For general χ, if T is a $\pi_\chi(\mathfrak{k})$-invariant distribution, then $T/\sigma_\chi = T\sigma_{-\chi}$ is $\pi_0(\mathfrak{k})$-invariant, and hence a constant. □

Proposition 2.4. *Let T be an M-invariant distribution on $\mathfrak{n}$ such that*

$$\pi_\chi(y + \theta y)T = 0 \textit{ for } \text{some } y \neq 0 \textit{ in } \overline{\mathfrak{n}},$$

then T is a multiple of the spherical vector σ_χ.

Proof. The M-invariance of T implies that

$$\pi_\chi(\mathfrak{m})T = 0$$

Since $\mathfrak{m}$ is a maximal subalgebra of $\mathfrak{k}$, $\mathfrak{m}$ and $y + \theta y$ generate $\mathfrak{k}$ as a Lie algebra. Thus

$$\pi_\chi(\mathfrak{k})T = 0,$$

and the result follows from the previous lemma. □

2.3. The K-invariance of the Bessel function. We now turn to the proof of Proposition 2.2. To simplify notation, we will write π instead of π_{χ_1}. Since Φ is clearly M-invariant, by Proposition 2.4 it suffices to show

$$\pi(y_1 + \theta y_1)\Phi = 0$$

for $y_1 \in \overline{\mathfrak{n}}$. We will prove this through a sequence of lemmas.

It is convenient to introduce the following notation: if g_1 and g_2 are functions on $\mathcal{O}_1$, we define

$$(g_1, g_2) = \int_{\mathcal{O}_1} g_1(y)\, g_2(y) d\mu_1(y)$$

provided the integral converges.

If g is a function on $\mathcal{O}_1$ and $h \in \mathfrak{l}$, then the action of h on g is given by

$$h \cdot g(y) \overset{\text{def}}{=} \frac{d}{dt} g(e^{th} \cdot y)\bigg|_{t=0}.$$

In the computation below, we shall work with the expressions of the type

$$\left(\frac{d}{dt} \int_{\mathcal{O}_1} g(e^{th} \cdot y) d\mu(y)\right)\bigg|_{t=0}.$$

To justify differentiation under the integral sign, we need to impose the standard conditions on g (e.g. [Ke, p.170]), as follows.

Define a class of functions $\mathcal{I} \subset C^\infty(\mathcal{O}_1)$, given by the following conditions: a smooth function g belongs to $\mathcal{I}$ if

- $g \in L^1(\mathcal{O}_1, d\mu_1)$ and
- for any $h \in \mathfrak{l}$ we can find $c > 0$ and $G(y) \in L^1(\mathcal{O}_1, d\mu_1)$, such that

$$\left| \frac{d}{dt} g(e^{th} \cdot y)\bigg|_{t=t_0} \right| \le G(y)$$

for all $y \in \mathcal{O}_1$ and $|t_0| < c$.

Lemma 2.5. *Suppose g_1, g_2 are smooth functions on $\mathcal{O}_1$, such that $g_1 g_2 \in \mathcal{I}$. Then*

$$(h \cdot g_1, g_2) + (g_1, h \cdot g_2) = 2d\nu(h)(g_1, g_2). \tag{6}$$

Proof. Using the L-equivariance of $d\mu_1$, we obtain

$$\int_{\mathcal{O}_1} g_1(e^{th} y) g_2(e^{th} y)\, d\mu_1 = e^{2td\nu(h)} \int_{\mathcal{O}_1} g_1 g_2\, d\mu_1.$$

Under the assumptions of the lemma, we can differentiate this identity in t, to get

$$\int_{\mathcal{O}_1} h \cdot (g_1 g_2)\, d\mu_1 = 2d\nu(h) \int_{\mathcal{O}_1} g_1 g_2\, d\mu_1.$$

By the Leibnitz rule, the result follows. □

More generally, if g_1, g_2 are functions on $\mathfrak{n} \times \mathcal{O}_1$, then (g_1, g_2) is a function on $\mathfrak{n}$. In this notation, for g in $L^1(\mathcal{O}_1, d\mu_1)$, the Fourier transform of $gd\mu_1$ is given by the formula

$$\widehat{gd\mu_1} = (e^{-i\langle x,y\rangle}, g).$$

Lemma 2.6. *Let $g \in L^1(\mathcal{O}_1, d\mu_1)$ be a smooth function on $\mathcal{O}_1$, such that*

$$e^{-i\langle x,y\rangle} g \in \mathcal{I}.$$

Suppose $f = (e^{-i\langle x,y\rangle}, g)$, then

$$\pi(y_1) f = -\frac{1}{2}(e^{-i\langle x,y\rangle}, h \cdot g(y)), \text{ where } h = [x, y_1].$$

Proof. By the formula for the action of $\pi(y_1)$, we get

$$\begin{aligned} -2\,(\pi(y_1) f + d\nu(h) f) &= \xi\,([h, x]) \cdot (e^{-i\langle x,y\rangle}, g) \\ &= \frac{d}{dt}\left(e^{-i\langle x+t[h,x],y\rangle}, g\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(e^{-i\langle x,y-t[h,y]\rangle}, g\right)\Big|_{t=0} \\ &= -\left(h \cdot e^{-i\langle x,y\rangle}, g\right) \\ &= (e^{-i\langle x,y\rangle}, h \cdot g) - 2d\nu(h)(e^{-i\langle x,y\rangle}, g). \end{aligned}$$

where we have used the previous lemma, and the relation

$$\begin{aligned} \langle x + t[h, x], y\rangle &= \langle x, y\rangle + t\,\langle [h, x], y\rangle \\ &= \langle x, y\rangle - t\,\langle x, [h, y]\rangle = \langle x, y - t[h, y]\rangle\,. \end{aligned}$$

The result follows. □

The pairing $-\langle \cdot, \theta \cdot\rangle$ gives a positive definite M-invariant inner product on $\overline{\mathfrak{n}}$, and we now obtain the following

Lemma 2.7. *Suppose that $g(y) = \phi\,(-\langle y, \theta y\rangle)$ for some smooth ϕ on $(0, \infty)$, and $e^{-i\langle x,y\rangle} g \in \mathcal{I}$. Put $f(x) = (e^{-i\langle x,y\rangle}, g)$, as before. Then*

$$\pi(y_1) f = \left(e^{-i\langle x,y\rangle}, \langle x, [[\theta y, y_1], y]\rangle\, \phi'\,(-\langle y, \theta y\rangle)\right).$$

Proof. Writing $h = [x, y_1]$ as in the previous lemma, we get

$$\begin{aligned} h \cdot g(y) &= \frac{d}{dt}\, \phi\,(-\langle y + t[h, y], \theta(y + t[h, y])\rangle)|_{t=0} \\ &= \frac{d}{dt}\, \phi\left(-\langle y, \theta y\rangle - 2t\,\langle \theta y, [h, y]\rangle + O(t^2)\right)\Big|_{t=0} \\ &= -2\,\langle \theta y, [h, y]\rangle\, \phi'\,(-\langle y, \theta y\rangle)\,. \end{aligned}$$

Since

$$\langle \theta y, [h, y]\rangle = \langle \theta y, [[x, y_1], y]\rangle = \langle x, [[\theta y, y_1], y]\rangle\,,$$

the result follows. □

The key lemma is the following computation

Lemma 2.8. *Let ϕ and f be as in the previous lemma, and suppose for $x \in \mathfrak{n}$*

$$e^{-i\langle x,y\rangle}\phi\,(-\langle y, \theta y\rangle) \in \mathcal{I},\quad e^{-i\langle x,y\rangle}\phi'\,(-\langle y, \theta y\rangle) \in \mathcal{I}. \tag{7}$$

Then we have

$$\pi(y_1 + \theta y_1)\, f(x) = \left(e^{-i\langle x,y\rangle}, i\,\langle \theta y_1, y\rangle\;(D\phi)\,(-\langle y, \theta y\rangle)\right), \tag{8}$$

where the differential operator D is given by the formula (2), *i.e.*

$$(D\phi)\,(-\langle y, \theta y\rangle) = 4\,(-\langle y, \theta y\rangle)\,\phi'' + 2(d + 1 - e)\phi' - \phi. \tag{9}$$

Proof. Choose a basis l_j of $\mathfrak{l}$ and define functions $c_j(y)$ by the formula $[\theta y, y_1] = \sum_j c_j(y) l_j$. Then by the previous lemma

$$\begin{aligned} \pi(y_1)\, f &= \sum_j (e^{-i\langle x,y\rangle}, \left\langle x, [l_j, y]\right\rangle c_j\phi') = i \sum_j \frac{d}{dt}\left(e^{-i\left\langle x, y + t[l_j, y]\right\rangle}, c_j\phi'\right)\Big|_{t=0} \\ &= i \sum_j (l_j \cdot e^{-i\langle x,y\rangle}, c_j\phi'). \end{aligned}$$

Differentiation in this calculation is justified, because $e^{-i\langle x,y\rangle}\phi'(-\langle y, \theta y\rangle) \in \mathcal{I}$.

Applying (6) to the last expression, we can write

$$\pi(y_1)\, f = -i \sum_j \left(e^{-i\langle x,y\rangle}, -2d\nu(l_j)c_j\phi' + c_j l_j \cdot \phi' + \phi' l_j \cdot c_j\right). \tag{10}$$

We now calculate each term in this expression.

- First we have

$$\sum_j \nu(l_j)c_j\phi' = \nu([\theta y, y_1])\phi'.$$

Since ν is a real character of $\mathfrak{l}$, it vanishes on $\mathfrak{l} \cap \mathfrak{k}$ and we have $\nu([\theta y, y_1]) = \nu([\theta y_1, y])$. Recall that $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ are irreducible $\mathfrak{l}$-modules. Therefore, $\nu([\theta y_1, y]) = k\langle\theta y_1, y\rangle$ for some constant $k \neq 0$, independent of y. Setting $y = y_1$, we get $\langle\theta y_1, y_1\rangle = \langle -x_1, y_1\rangle = -1$. Hence $k = -\nu([\theta y_1, y_1]) = 1$, and therefore

$$-\sum_j 2d\nu(l_j)c_j\phi' = -2d\langle\theta y_1, y\rangle\phi'. \tag{11}$$

- Next we compute

$$\begin{aligned}\sum_j c_j l_j \cdot \phi' &= [\theta y, y_1]\cdot\phi' \\ &= \frac{d}{dt}\,\phi'\left(-\langle y + t[[\theta y, y_1], y], \theta(y + t[[\theta y, y_1], y])\rangle\right)\Big|_{t=0} \\ &= -2\langle y, [[y, \theta y_1], \theta y]\rangle\,\phi''\left(-\langle y, \theta y\rangle\right)\end{aligned}$$

Since y is a $\mathfrak{k}$-conjugate to a root vector, there is a scalar k' independent of y such that $[[y, \theta y], y] = k'\langle y, \theta y\rangle y$. Setting $y = y_1$ we get $\langle y_1, \theta y_1\rangle = -1$,

$$[[y_1, -x_1], y_1] = -2y_1$$

and $k' = 2$. Also $-\langle y, [[y, \theta y_1], \theta y]\rangle = \langle [y, \theta y], [y, \theta y_1]\rangle = \left\langle [[y, \theta y], y], \theta y_1\right\rangle$. Hence

$$\sum_j c_j l_j \cdot \phi' = 4\langle y, \theta y\rangle\langle\theta y_1, y\rangle\phi''. \tag{12}$$

- Next we note that $\sum_j l_j \cdot c_j$ is independent of the basis l_j, so we may assume that

$$\theta l_j = \pm l_j \text{ and } \left\langle l_j, -\theta l_k\right\rangle = \delta_{jk}.$$

Then $c_j(y) = \left\langle [\theta y, y_1], -\theta l_j\right\rangle$ and

$$\begin{aligned}\textstyle\sum_j l_j \cdot c_j &= \textstyle\sum_j \left\langle [\theta[l_j, y], y_1], -\theta l_j\right\rangle \\ &= \textstyle\sum_j \left\langle y_1, [\theta[l_j, y], \theta l_j]\right\rangle = -\langle y_1, \Omega\theta y\rangle .\end{aligned}$$

Here $\Omega = \sum_j \mathrm{ad}(\theta l_j)^2 = \Omega_{\mathfrak{l}} - 2\Omega_{\mathfrak{k}\cap\mathfrak{l}}$, where the Casimir elements are obtained by using dual bases with respect to $\langle\ ,\ \rangle$.

To continue, we need the following lemma:

Lemma 2.9. *Ω acts on $\mathfrak{n}$ by the scalar $k'' = 2 - 2e$.*

Proof. When $e = 1$ it's easy to see that the operator Ω acts by 0. Indeed, in this case $\mathfrak{g}$ is a complex semisimple Lie algebra and for each basis element $l_j \in \mathfrak{k} \cap \mathfrak{l}$ there exists a basis element $l'_j = \sqrt{-1} l_j \in \mathfrak{p} \cap \mathfrak{l}$. Then $[l_j, [l_j, x]] + [l'_j, [l'_j, x]] = 0$ and $k'' = 0$.

When $e = 0$, $\mathfrak{g}$ is split and simply laced, and $\mathfrak{l}$ is the split real form of a complex reductive algebra $\mathfrak{l}_{\mathbb{C}}$. Take a root vector $x_\lambda \in \mathfrak{g}_\lambda$, where λ is any positive root in $\mathfrak{n}$. For any positive root α of $\mathfrak{l}_{\mathbb{C}}$ we fix $e_\alpha \in \mathfrak{l}_\alpha$ and set $l_\alpha = e_\alpha + \theta e_\alpha \in \mathfrak{k} \cap \mathfrak{l}$ and $l'_\alpha = e_\alpha - \theta e_\alpha \in \mathfrak{p} \cap \mathfrak{l}$. Then the collection of all l_α, l'_α together with the orthonormal basis of a Cartan subalgebra $\mathfrak{f}$ of $\mathfrak{l}$ forms a basis of $\mathfrak{l}$. Observe that

$$[l_\alpha, [l_\alpha, x_\lambda]] + [l'_\alpha, [l'_\alpha, x_\lambda]] = [e_\alpha, [e_\alpha, x_\lambda]] + [e_{-\alpha}, [e_{-\alpha}, x_\lambda]] = 0,$$

since $x_\lambda \in \mathfrak{g}_\lambda$ and neither $\lambda + 2\alpha$ nor $\lambda - 2\alpha$ is a root of the simply laced algebra $\mathfrak{g}_{\mathbb{C}}$.

We choose a basis $\{u_i\}$ of $\mathfrak{f}$, and denote the elements of the dual (with respect to $\langle\ ,\ \rangle$) basis by $\widetilde{u}_i$. Then

$$\Omega x_\lambda = \sum_i [u_i, [\widetilde{u}_i, x_\lambda]] = \langle \lambda, \lambda \rangle\, x_\lambda = 2x_\lambda.$$

In the remaining two cases $\mathfrak{k} \cap \mathfrak{l}$ acts on $\mathfrak{n}$ irreducibly, therefore Ω automatically acts by a scalar and it suffices to compute $\sum_j [l_j, [l_j, x_1]]$. For $e = 3$ we have $G = GL_{2n}(\mathbb{H})$, $L = GL_n(\mathbb{H}) \times GL_n(\mathbb{H})$ and $\mathfrak{n} = \mathbb{H}^{n\times n}$. The computation for this group is similar to the case of $G = GL_{2n}(\mathbb{R})$. We reduce the calculation to the summation over the diagonal subalgebra of $\mathfrak{l}$ and obtain

$$\Omega x_\lambda = \langle \lambda, \lambda \rangle\, x_\lambda + 3 \left\langle \sqrt{-1}\lambda, \sqrt{-1}\lambda \right\rangle x_\lambda = -4x_\lambda.$$

Finally, for $e = 2$ ($G = Sp_{n,n}$), a direct evaluation of $\sum_j [l_j, [l_j, x_1]]$ gives $k'' = -2$. □

Therefore, we get

$$\sum_j \phi' l_j \cdot c_j = -2(1-e)\, \langle \theta y_1, y \rangle\, \phi'. \tag{13}$$

- Finally, we have

$$\pi(\theta y_1)\, f = \frac{d}{dt} \left(e^{-i\langle x + t\theta y_1, y\rangle}, \phi \right)\Big|_{t=0} = -i \left(e^{-i\langle x, y\rangle}, \langle \theta y_1, y \rangle\, \phi \right). \tag{14}$$

Putting the formulas (11)–(14) together, we deduce the lemma. □

Proof of Proposition 2.2. Recall that we study $\phi_\tau(z) = \dfrac{K_\tau(\sqrt{z})}{\left(\sqrt{z}\right)^\tau}$, its lift g_τ to the radial function on $\mathcal{O}_1$,

$$g_\tau(y) = \phi_\tau(-\langle y, \theta y\rangle) = \frac{K_\tau(|y|)}{|y|^\tau}$$

and its Fourier transform $\Phi(x) = (e^{-i\langle x,y\rangle}, g_\tau)$. By Proposition 2.4 it suffices to check that $\pi(y_1 + \theta y_1)\Phi = 0$. This identity would follow immediately from Lemma 2.8, because $D\phi_\tau = 0$ by formula (2) and then the desired result follows from (8).

To complete the proof we have to verify the assumptions (7). In Subsection 2.1 we proved that $g_\tau \in L^1(\mathcal{O}_1, d\mu_1)$. It is easy to verify (using the standard facts about the derivatives of K_τ from [W]), that the lifts to $\mathcal{O}_1$ of the functions $\phi'_\tau(z)$ and $\phi''_\tau(z)$ (we denote them by $g'_\tau(y)$ and $g''_\tau(y)$) both belong to $L^1(\mathcal{O}_1, d\mu_1)$. Observe also that $\phi_\tau(z)$, $\phi'_\tau(z)$, $\phi''_\tau(z)$ are all monotone on $(0, \infty)$.

Moreover, since all these functions tend to zero exponentially as $|y| \to \infty$, the functions $A(y)g_\tau(y)$, $A(y)g'_\tau(y)$, $A(y)g''_\tau(y)$ all belong to $L^1(\mathcal{O}_1, d\mu_1)$, for any $A(y)$ bounded in the neighbourhood of $y = 0$ and growing (at most) polynomially with respect to $|y|$ as $|y| \to \infty$.

Fix $h \in \mathfrak{l}$, $x \in \mathfrak{n}$ and choose $c > 0$ sufficiently small, such that for all $y \in \mathcal{O}_1$ and $|t| < c$

$$\left|\left\langle e^{th} \cdot y, \theta(e^{th} \cdot y)\right\rangle\right| \geq \frac{|\langle y, \theta y\rangle|}{2}.$$

We can then estimate the derivative:

$$\left|\frac{d}{dt}\left(e^{-i\left\langle x, e^{th}\cdot y\right\rangle}\phi_\tau\left(-\left\langle e^{th}y, \theta e^{th}y\right\rangle\right)\right)\right|$$
$$\leq \left|A_1(y)\phi_\tau\left(|y|^2/2\right)\right| + \left|A_2(y)\phi'_\tau\left(|y|^2/2\right)\right|,$$

for all $y \in \mathcal{O}_1$ and $|t| < c$, where $A_1(y), A_2(y)$ are some functions of polynomial growth. From the discussion above, the right-hand side of this inequality is an L^1-function on $\mathcal{O}_1$, hence $e^{-i\langle x,y\rangle}g_\tau \in \mathcal{I}$.

Proceeding in the same manner, we deduce that $e^{-i\langle x,y\rangle}g'_\tau \in \mathcal{I}$. □

2.4. Proof of Theorem 0.1. Denote by **J** the space of the induced representation $\pi_1 = \mathrm{Ind}_P^G(e^{-d\nu})$. By the Gelfand-Naimark decomposition and the exp map, **J** can be viewed as a subspace of $C^\infty(\mathfrak{n})$. Then for $l \in L$ and $\eta \in$ **J** we have

$$\pi_1(l)\eta(x) = e^{-d\nu}(l)\eta(l^{-1} \cdot x).$$

It was proved in [S3] that the $(\mathfrak{g}, K)$-module $\mathbf{J}$ has a unitarizable spherical $(\mathfrak{g}, K)$-submodule V, which we also regard as a subspace of $C^\infty(\mathfrak{n})$.

Remark. It is possible to give a direct description of the elements of the "abstract" Hilbert space $\mathcal{H}$, where $\mathcal{H}$ is the Hilbert space closure of V with respect to the $(\mathfrak{g}, K)$-invariant norm on V. For that purpose we use the "compact" realization of π_1 on $C^\infty(K/M)$ from [S3]. It was shown that π_1 is a representation of ladder type, with all its K-types $\{\alpha_m \mid m \in \mathbb{N}\}$ lying on a single line, α_1 being a one-dimensional K-type. The restriction $\langle\ ,\ \rangle_m$ of a π_1-invariant Hermitian form to any K-type α_m is a multiple of the $L^2(K)$-inner product on V, and from the explicit formulas in [S3] it follows that

$$q_m \overset{\text{def}}{=} \frac{\langle\ ,\ \rangle_m}{\langle\ ,\ \rangle_1} = O(m^C)$$

for some constant $C > 1$, which can be expressed in terms of parameters d, e and n. Thus we can identify $\mathcal{H}$ with the Hilbert space $L^2(\mathbb{N}, \{q_m\})$, where the constant q_m gives the weight of the point $m \in \mathbb{N}$. That is, any element of $\mathcal{H}$ can be viewed as an M-equivariant function on K, such that its sequence of Fourier coefficients belongs to $L^2(\mathbb{N}, \{q_m\})$. In particular $L^2(\mathbb{N}, \{q_m\}) \subset l^2(\mathbb{N})$, and the elements of $\mathcal{H}$ all lie in $L^2(K)$.

We write $\mathbf{H}$ for the space of those tempered distributions on $\mathfrak{n}$ which are Fourier transforms of $\psi d\mu_1$ for some $\psi \in L^2(\mathcal{O}_1, d\mu_1)$. If η is the Fourier transform of a distribution of the form $\psi d\mu_1$, i.e.,

$$\eta(x) = \int_{\mathcal{O}_1} e^{-i\langle x,y\rangle}\psi(y)d\mu_1(y) = \left(e^{-i\langle x,y\rangle}, \psi(y)\right),$$

then

$$\begin{aligned}\pi_1(l)\eta(x) &= e^{-d\nu}(l)\eta(l^{-1}\cdot x) = \int_{\mathcal{O}_1} e^{-i\langle l^{-1}\cdot x,y\rangle}\psi(y)e^{-d\nu}(l)d\mu_1(y)\\ &= \int_{\mathcal{O}_1} e^{-i\langle l^{-1}x,l^{-1}y\rangle}\psi(l^{-1}\cdot y)e^{-d\nu}(l)d\mu_1(l^{-1}\cdot y)\\ &= (e^{-i\langle x,y\rangle}, e^{d\nu}(l)\psi(l^{-1}\cdot y)).\end{aligned}$$

It follows from the calculation above that P acts *unitarily* on $\mathbf{H}$ (it is convenient to identify this action with its realization on $L^2(\mathcal{O}_1, d\mu_1)$ via the Fourier transform).We denote this unitary representation of P by π'. Observe that $(\pi', \mathbf{H})$ is an irreducible representation of P.

According to Proposition 2.1, $\Phi(x) = (e^{-i\langle x,y\rangle}, |y|^{-\tau} K_\tau(|y|))$ belongs to $\mathbf{H}$.

Theorem 2.10. *V is a dense subspace of $\mathbf{H}$, and the restriction of the norm is $(\mathfrak{g}, K)$-invariant.*

Proof. Let $C^\infty(K)_V$ be the subspace of $C^\infty(K)$, consisting of those smooth functions on K, whose K-isotypic components belong to V. Since V is a submodule of **J**, $C^\infty(K)_V$ is obviously G-invariant.

Denote by $\mathcal{C}(G)$ the convolution algebra of smooth L^1 functions on $G = PK$, and consider

$$\mathbf{W} = \pi_1(\mathcal{C}(G))\Phi \subset C^\infty(K)_V.$$

So all elements of **W** are continuous functions on K, hence continuous on G, and therefore are determined by their restrictions to N. Moreover, $\mathbf{W} = \pi_1(\mathcal{C}(PK))\Phi$ and K fixes Φ, therefore

$$\mathbf{W} = \pi_1(\mathcal{C}(P))\Phi = \pi'(\mathcal{C}(P))\Phi.$$

This shows that **W** is a $\pi'(P)$-invariant subspace of **H**, and from the irreducibility of π' we conclude that **W** is dense in **H**.

We can now put two $\pi_1(P)$-invariant norms on **W** – one from **H** and another from V, as follows. If $f = \sum c_m v_m$, with v_m in the K-isotypic component with highest weight α_m (occurring in V) and $\|v_m\|_{L^2(K)} = 1$, then

$$\|f\|_V^2 = \sum |c_m|^2 q_m. \tag{15}$$

Since f is smooth, it follows that $|c_m|$ decays rapidly, so the series in (15) converges, thus giving a $\pi_1(P)$-invariant norm on **W**.

Then it follows from [P] (cf. [S1, p.417]), that we can find a (dense) $\mathcal{C}(P)$-invariant subspace $\mathbf{W}' \subset \mathbf{W}$, such that these two forms are proportional on $\mathbf{W}'$. Considering the closure of $\mathbf{W}'$ we obtain an isometric P-invariant imbedding of **H** into $\mathcal{H}$.

Then **W** is:

(1) a G-invariant subspace of the irreducible module $\mathcal{H}$, hence dense in $\mathcal{H}$;

(2) a dense subspace of the Hilbert space **H**.

It follows that $\mathbf{H} = \mathcal{H}$. □

This concludes the proof of Theorem 0.1.

3. Tensor powers of π_1

3.1. Restrictions to P. In the previous section we constructed a unitary representation π_1 of G acting on the Hilbert space $L^2(\mathcal{O}_1, d\mu_1)$, where $\mathcal{O}_1$ is the minimal L-orbit in a non-Euclidean Jordan algebra N. Define the k-th tensor power representation

$$\Pi_k = \pi_1^{\otimes k} (2 \leq k < n).$$

As we shall show, the techniques developed in [DS] allow us to establish a duality between the spectrum of this tensor power and the spectrum of a certain homogeneous space. We omit the proofs of the several propositions below, because the proofs of the corresponding statements from [DS] can be used without any substantial modification.

Observe that the orbit $\mathcal{O}_k$ is dense in $\underbrace{\mathcal{O}_1 + \mathcal{O}_1 + \ldots + \mathcal{O}_1}_{k \text{ times}}$. The representation Π_k acts on $\left[L^2(\mathcal{O}_1, d\mu_1)\right]^{\otimes k} \simeq L^2(\mathcal{O}'_k, d\mu')$, where $\mathcal{O}'_k = \mathcal{O}_1^{\times k}$ and $d\mu'$ is the product measure on $\mathcal{O}'_k$. We fix a generic representative $\xi' = (\xi_1, \xi_2, \ldots, \xi_k) \in \mathcal{O}'_k$, such that

$$\xi = \xi_1 + \xi_2 + \ldots + \xi_k \in \mathcal{O}_k.$$

Denote by S'_k and S_k the isotropy subgroups of ξ' and ξ, respectively, with respect to the action of L on $\mathcal{O}'_k$ and $\mathcal{O}_k$. Observe that the Lie algebras $\mathfrak{s}'_k$ and $\mathfrak{s}_k$ of S'_k and S_k, respectively, can be written as

$$\mathfrak{s}'_k = (\mathfrak{h}_k + \mathfrak{l}_k) + \mathfrak{u}_k$$
$$\mathfrak{s}_k = (\mathfrak{g}_k + \mathfrak{l}_k) + \mathfrak{u}_k.$$

Here $\mathfrak{l}_k$, $\mathfrak{g}_k$ and $\mathfrak{h}_k$ are reductive, $\mathfrak{h}_k \subset \mathfrak{g}_k$ and $\mathfrak{u}_k$ is a nilpotent radical common for both $\mathfrak{s}'_k$ and $\mathfrak{s}_k$. Let G_k and H_k be the corresponding Lie groups.

Example. Take $G = O_{2n,2n}$ and $k < n$. Then $\xi_i = E_{2i-1,2i} - E_{2i,2i-1}$ $(1 \le i \le k)$, $\xi = \sum_{i=1}^k \xi_i$ and

$$\mathfrak{s}_k = \left(\mathfrak{sp}_{2k}(\mathbb{R}) + \mathfrak{gl}_{2(n-k)}(\mathbb{R})\right) + \mathbb{R}^{2k,2(n-k)}.$$

Then $G_k = Sp_{2k}(\mathbb{R})$ and it's easy to check that $H_k = SL_2(\mathbb{R})^k$.

The following Lemma can be verified by direct calculation (cf. [DS, Lemma 2.1]).

Lemma 3.1. *Let χ_ξ be the character of N corresponding to $\xi \in N^*$. Then*

$$\Pi_k|_P = \mathrm{Ind}_{S'_k N}^P (1 \otimes \chi_\xi) = \mathrm{Ind}_{S_k N}^P \left(\left(\mathrm{Ind}_{S'_k}^{S_k} 1\right) \otimes \chi_\xi\right) \quad (L^2\text{-induction}).$$

□

Let $\gamma' = \mathrm{Ind}_{H_k}^{G_k} 1$ be the quasiregular representation of G_k on $L^2(G_k/H_k)$; then it can be decomposed using the Plancerel measure $d\mu$ for the reductive homogeneous space $X_k = G_k/H_k$ and the corresponding multiplicity function $m : \widehat{G}_k \to \{0, 1, 2, \ldots\}$, i.e.,

$$\gamma' \simeq \int_{\widehat{G}_k}^{\oplus} m(\kappa)\kappa \, d\mu(\kappa).$$

Each irreducible representation κ of G_k can be extended to an irreducible representation $\kappa^\vee$ of S_k, and the decomposition of the Lemma above can be rewritten as

$$\Pi_k|_P = \int_{\widehat{G}_k}^{\oplus} m(\kappa)\Theta(\kappa)\, d\mu(\kappa), \tag{16}$$

where $\Theta(\kappa) = \mathrm{Ind}_{S_k N}^{P}(\kappa^{\vee} \otimes \chi_{\xi})$.

Moreover, by Mackey theory all representations $\Theta(\kappa)$ are unitary irreducible representations of P, and $\Theta(\kappa') \simeq \Theta(\kappa'')$ if and only if $\kappa' \simeq \kappa''$.

3.2. Low-rank theory. In [DS] we extended the theory of low-rank representations ([Li]) to the conformal groups of euclidean Jordan algebras. Inspection of the argument in [DS] shows that the analogous theory can be developed in exactly the same manner for the conformal groups of *non-euclidean* Jordan algebras.

For any unitary representation η of G, we decompose its restriction $\eta|_N$ into a direct integral of unitary characters, where the decomposition is determined by a projection-valued measure on $\widehat{N} = N^*$. If this measure is supported on a single *non-open* L-orbit $\mathcal{O}_m, 1 \le m < n$ we call η a *low-rank representation*, and write rank $\eta = m$. Proceeding by induction on m, as in [Li], [DS, Sect 3], we can prove the following

Theorem 3.2. *Let η be a low-rank representation of G. Write $\mathcal{A}(\eta, P)$ for the von Neumann algebra generated by $\{\eta(x) | x \in P\}$ and $\mathcal{A}(\eta, G)$ for the von Neumann algebra generated by $\{\eta(x) | x \in G\}$. Then $\mathcal{A}(\eta, G) = \mathcal{A}(\eta, P)$.* □

Proof of Theorem 0.2. Now consider the restriction of Π_k to N. Its restriction to P is given by the direct integral decomposition (16), and we can further restrict it to N. The rank of the induced representation $\Theta(\kappa) = \mathrm{Ind}_{S_k N}^{P}(\kappa^{\vee} \otimes \chi_{\xi})$ is k (the N–spectrum is supported on the L−orbit of ξ, i.e. $\mathcal{O}_k$). Therefore Π_k can be decomposed over the irreducible representations of G of rank k.

It follows from the theorem above that any two non-isomorphic representations from the spectrum of Π_k restrict to non-isomorphic irreducible representations of P. Hence the representation Π_k can be decomposed as

$$\Pi_k = \int_{\widehat{G}_k}^{\oplus} m(\kappa)\theta(\kappa)\, d\mu(\kappa), \tag{17}$$

where for almost every κ the unitary irreducible representation $\theta(\kappa)$ is obtained as the *unique* irreducible representation of G determined by the condition $\theta(\kappa)|_P = \Theta(\kappa)$.

Therefore, the map $\kappa \to \theta(\kappa)$ gives a (measurable) bijection between the spectrum of $\Pi_k = \pi^{\otimes k}$ and the unitary representations of G_k occurring in the quasiregular representation on $L^2(G_k/H_k)$. □

Example. Take $G = E_{7(7)}$. It is the conformal group of the split exceptional real Jordan algebra N of dimension 27. Consider the tensor square of the minimal representation π_1 of G ($k = 2$). Then $L = \mathbb{R}^* \times E_{6(6)}$, S_2' is the stabilizer of y_1 **and** y_2 and S_2 is the stabilizer of $y_1 + y_2 \in \mathcal{O}_2$. One can see that in this case $\mathfrak{g}_2 = \mathrm{Stab}_{\mathfrak{so}(5,5)}(y_1 + y_2) = \mathfrak{so}(4,5)$ and $\mathfrak{h}_2 = \mathrm{Stab}_{\mathfrak{so}(5,5)}(y_1) \cap \mathrm{Stab}_{\mathfrak{so}(5,5)}(y_2) = \mathfrak{so}(4,4)$ (cf. [A, 16.7]). Hence the decomposition (17) establishes a duality between the representations of $E_{7(7)}$ occurring in $\Pi_2 = \pi_1 \otimes \pi_1$ and the unitary representations of $Spin(4,5)$ occurring in $L^2\,(Spin(4,5)/Spin(4,4))$. The homogeneous space $Spin(4,5)/Spin(4,4)$ is a (pseudo-riemannian) symmetric space of rank 1, and it is known to be multiplicity free. Therefore, $\pi_1 \otimes \pi_1$ has simple spectrum.

Similarly, for $G = E_7(\mathbb{C})$ we obtain a duality between $E_7(\mathbb{C})$ and the symmetric space $SO_9(\mathbb{C})/SO_8(\mathbb{C})$.

A. Groups associated to non-Euclidean Jordan algebras

G	K/M	d	e	G_k/H_k for $2 \le k < n$
$GL_{2n}(\mathbb{R})$	$O_{2n}/(O_n \times O_n)$	1	0	$GL_k(\mathbb{R})/[GL_1(\mathbb{R})]^k$
$O_{2n,2n}$	$(O_{2n} \times O_{2n})/O_{2n}$	2	0	$Sp_{2k}(\mathbb{R})/[SL_2(\mathbb{R})]^k$
$E_{7(7)}$	SU_8/Sp_4	4	0	$Spin(4,5)/Spin(4,4)$
$O_{p+2,p+2}$	$[O_{p+2}]^2/[O_1 \times O_{p+1}^2]$	p	0	
$Sp_n(\mathbb{C})$	Sp_n/U_n	1	1	$O_k(\mathbb{C})/[O_1(\mathbb{C})]^k$
$GL_{2n}(\mathbb{C})$	$U_{2n}/(U_n \times U_n)$	2	1	$GL_k(\mathbb{C})/[GL_1(\mathbb{C})]^k$
$O_{4n}(\mathbb{C})$	O_{4n}/U_{2n}	4	1	$Sp_{2k}(\mathbb{C})/[SL_2(\mathbb{C})]^k$
$E_7(\mathbb{C})$	$E_7/(E_6 \times U_1)$	8	1	$SO_9(\mathbb{C})/SO_8(\mathbb{C})$
$O_{p+4}(\mathbb{C})$	$O_{p+4}/(O_{p+2} \times U_1)$	p	1	
$Sp_{n,n}$	$(Sp_n \times Sp_n)/Sp_n$	2	2	$O_k^*/[O_1^*]^k$
$GL_{2n}(\mathbb{H})$	$Sp_{2n}/(Sp_n \times Sp_n)$	4	3	$GL_k(\mathbb{H})/[GL_1(\mathbb{H})]^k$

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