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THE BINOMIAL FORMULA FOR NONSYMMETRIC MACDONALD POLYNOMIALS

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1. Introduction. The q-binomial theorem [GR] is essentially the expansion of $(x-1)(x-q)\cdots(x-q^{k-1})$ in terms of the monomials x^d. In a recent paper [Ok], Okounkov has proved a beautiful multivariate generalization of this in the context of symmetric Macdonald polynomials [M1]. These polynomials have nonsymmetric counterparts [M2] that are of substantial interest; in this paper, we establish nonsymmetric analogues of Okounkov's results.

An integral vector $v \in \mathbb{Z}^n$ is called "dominant" if $v_1 \geqslant \cdots \geqslant v_n$; it is called a "composition" if $v_i \geqslant 0$ for all i. To avoid ambiguity, we reserve the letters u, v for integral vectors, α, β, γ for compositions, and λ, μ for "partitions" (dominant compositions).

We write $|v|$ for $v_1 + \cdots + v_n$, and denote by w_v the (unique) shortest permutation in the symmetric group S_n such that $v^+ = w_v^{-1}(v)$ is dominant. Let $\mathbb{F}$ be the field $\mathbb{Q}(q,t)$ where q,t are indeterminates. We write $\tau = (1, t^{-1}, \ldots, t^{-n+1})$ and define $\bar{v} = \bar{v}(q,t)$ in $\mathbb{F}^n$ by

$$\bar{v}_i = q^{v_i}(w_v\tau)_i.$$

Inhomogeneous analogues of nonsymmetric Macdonald polynomials were introduced in [Kn] and [S3]. They form an $\mathbb{F}$-basis for $\mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]$, and are defined as follows.

Definition. $G_\alpha \equiv G_\alpha(x; q, t)$ is the unique polynomial of degree $\leqslant |\alpha|$ in $\mathbb{F}[x]$ such that

(1) the coefficient of $x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in G_α is 1;
(2) G_α vanishes at $x = \bar{\beta}$, for all compositions $\beta \neq \alpha$ such that $|\beta| \leqslant |\alpha|$.

As shown in Theorem 3.9 of [Kn], the top homogeneous part of G_α is the nonsymmetric Macdonald polynomial E_α for the root system A_{n-1} (see [M2] and [C]). Moreover, by Theorem 4.5 of [Kn], we have $G_\alpha(\bar{\beta}) = 0$ unless "$\alpha \subseteq \beta$." Here $\alpha \subseteq \beta$ means that if we write $w = w_\beta w_\alpha^{-1}$, then $\alpha_i < \beta_{w(i)}$ if $i < w(i)$ and $\alpha_i \leqslant \beta_{w(i)}$ if $i \geqslant w(i)$.

In this paper, we obtain several new results about the polynomials G_α. Our first result is a formula for the special value $G_\alpha(a\bar{0}) = G_\alpha(a\tau) \in \mathbb{F}[a]$, where a is an indeterminate. This can be described in the following manner. We identify α

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with the "diagram" consisting of points $(i,j) \in \mathbb{Z}^2$ with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant \alpha_i$. For $s = (i,j) \in \alpha$, we define the *arm, leg, coarm,* and *coleg* of s by

$$a(s) = \alpha_i - j, \quad l(s) = \#\{k > i | j \leqslant \alpha_k \leqslant \alpha_i\} + \#\{k < i | j \leqslant \alpha_k + 1 \leqslant \alpha_i\},$$

$$a'(s) = j - 1, \quad l'(s) = \#\{k > i | \alpha_k > \alpha_i\} + \#\{k < i | \alpha_k \geqslant \alpha_i\}.$$

THEOREM 1.1. *We have*

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left(\frac{t^{1-n} - q^{a'(s)+1} t^{1-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}} \right) \prod_{s \in \alpha} (a t^{l'(s)} - q^{a'(s)}).$$

Let w_o be the longest element of S_n (which interchanges each i with $n - i + 1$), and put $\tilde{\beta} = \overline{-w_o \beta}$ and $\bar{\beta}^{-1} = \bar{\beta}(q^{-1}, t^{-1}) = (\bar{\beta}_1^{-1}, \ldots, \bar{\beta}_n^{-1})$. Then we have the following crucial "reciprocity" result.

THEOREM 1.2. *There is a (unique) polynomial O_α of degree $\leqslant |\alpha|$ in $\mathbb{Q}(q,t,a)[x]$ such that $O_\alpha(\bar{\beta}^{-1}) = G_\beta(a\tilde{\alpha})/G_\beta(a\tau)$ for all β.*

We now introduce the following variants of G_α, which also form a basis for $\mathbb{F}[x]$.

Definition. $G'_\alpha = G'_\alpha(x; q, t)$ is the unique polynomial in $\mathbb{F}[x]$ such that
(1) G'_α and G_α have the same top-degree terms, that is, E_α;
(2) G'_α vanishes at $x = \tilde{\beta}$ for all β with $|\beta| < |\alpha|$.

The existence of G'_α can be proved along the same lines as that of G_α (see [Kn, Theorem 2.3] and [S3, Theorem 4.3]). One verifies that polynomials of degree $\leqslant d$ are uniquely determined by their values at $x = \tilde{\beta}$ for $|\beta| \leqslant d$. Hence the lower-degree terms of G'_α are determined by (2).

Definition. The "nonsymmetric (q,t)-binomial coefficients" are defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} = \frac{G_\beta(\bar{\alpha})}{G_\beta(\bar{\beta})} \equiv \frac{G_\beta(\bar{\alpha}(q,t); q,t)}{G_\beta(\bar{\beta}(q,t); q,t)}.$$

Our main result is the following relationship between G_α and G'_β.

THEOREM 1.3. *We have*

$$\frac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_{\beta \subseteq \alpha} a^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{1/q, 1/t} \frac{G'_\beta(x)}{G_\beta(a\tau)}.$$

COROLLARY 1.4. *We have*

$$\frac{G_\alpha(x)}{G_\alpha(0)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{1/q,1/t} \frac{E_\beta(x)}{G_\beta(0)}. \qquad \square$$

COROLLARY 1.5. *We have*

$$\frac{E_\alpha(x)}{E_\beta(\tau)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{1/q,1/t} \frac{G'_\beta(x)}{E_\beta(\tau)}. \qquad \square$$

The corollaries follow from Theorem 1.3 by: (1) replacing x by $a^{-1}x$ and letting $a \to 0$ and (2) letting $a \to \infty$. For $n = 1$, Corollary 1.4 is essentially the q-binomial theorem.

If we put $t = q^r$ and let $q \to 1$, then $E_\alpha(x;r) \equiv \lim_{q\to 1} E_\alpha(x;q,q^r)$ is the nonsymmetric Jack polynomial (see [Op]). To discuss this limiting case, we define $\delta \equiv (0,-1,\ldots,-n+1)$, $\rho = r\delta$, and $\bar{\alpha}(r) = \alpha + w_\alpha\rho$.

Definition. $G_\alpha(x;r)$ is the unique polynomial of degree $\leqslant |\alpha|$ in $\mathbb{Q}(r)[x]$ such that

(1) the coefficient of $x^\alpha \equiv x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ in $G_\alpha(x;r)$ is 1;
(2) $G_\alpha(x;r)$ vanishes at $x = \bar{\beta}(r)$ for all compositions $\beta \neq \alpha$ such that $|\beta| \leqslant |\alpha|$.

Definition. The "nonsymmetric r-binomial coefficients" are

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r = \frac{G_\beta(\bar{\alpha}(r);r)}{G_\beta(\bar{\beta}(r);r)}.$$

Definition. $G'_\alpha(x;r)$ is the unique polynomial in $\mathbb{Q}(r)[x]$ such that
(1) $G'_\alpha(x;r)$ and $G_\alpha(x;r)$ have the same top-degree terms;
(2) $G'_\alpha(x;r)$ vanishes at $x = \tilde{\beta}(r) \equiv \overline{-w_o\beta}(r)$ for all β with $|\beta| < |\alpha|$.

Theorems 1.1–1.3 have analogues in this setting.
If a is a scalar and x is a vector, write $a + x$ for $(a + x_1, \ldots, a + x_n)$.

THEOREM 1.6. *We have*

$$G_\alpha(a+\rho;r) = \prod_{s\in\alpha}\left(\frac{a'(s)+1-rl'(s)+rn}{a(s)+1+rl(s)+r}\right)\prod_{s\in\alpha}(a-a'(s)+rl'(s)).$$

THEOREM 1.7. *There is a (unique) polynomial* $O_\alpha(x;r)$ *of degree* $\leqslant |\alpha|$ *in* $\mathbb{Q}(a,r)[x]$ *such that we have* $O_\alpha(\bar{\beta}(r);r) = G_\beta(a+\tilde{\alpha}(r);r)/G_\beta(a+\rho;r)$ *for all* β.

THEOREM 1.8. *We have*

$$\frac{G_\alpha(a+x;r)}{G_\alpha(a+\rho;r)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \frac{G'_\beta(x;r)}{G_\beta(a+\rho;r)}.$$

Since $\bar{\alpha}_i(r) = \lim_{q\to1}(\bar{\alpha}_i(q,q^r)-1)/(q-1)$ as in [Kn, Theorem 6.2], we get $G_\alpha(x;r) \equiv \lim_{q\to1} G_\alpha(1+(q-1)x;q,q^r)/(q-1)^{|\alpha|}$. It follows that the top terms of $G_\alpha(x;r)$ and $G'_\alpha(x;r)$ are $E_\alpha(x;r)$, and that

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r = \lim_{q\to1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{q,q^r} = \lim_{q\to1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_{1/q,1/q^r}.$$

So setting $x = ax$ and letting $a \to \infty$ in Theorem 1.8, we get the following corollary.

COROLLARY 1.9. *We have*

$$\frac{E_\alpha(1+x;r)}{E_\alpha(1;r)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \frac{E_\beta(x;r)}{E_\beta(1;r)}. \qquad \square$$

It seems to be difficult to deduce Theorems 1.6–1.8 directly from Theorems 1.1–1.3 by a limiting procedure. However, the *proofs* in the (q,t)-case *can* be modified to make them work in this setting.

We now describe some *new* phenomena in the limiting case. Write s_i for the transposition $(i\ i+1)$, which acts on $\mathbb{Q}(a)[x]$ by permuting x_i and x_{i+1}, and let

$$\sigma_i = s_i + \frac{r}{x_i - x_{i+1}}(1-s_i).$$

Then, as observed in [Kn, Corollary 6.5], the map $\sigma: s_i \mapsto \sigma_i$ extends to a representation of S_n.

THEOREM 1.10. *We have* $G'_\alpha(x;r) = (-1)^{|\alpha|}\sigma(w_o)w_o G_\alpha(-x-(n-1)r;r)$.

Using this and writing $G^+_\alpha(x;r) := (-1)^{|\alpha|}G_\alpha(-x-(n-1)r;r)$, we get the following corollary.

COROLLARY 1.11. *We have*

$$\frac{\sigma(w_o)G_\alpha(a+x;r)}{G_\alpha(a+\rho;r)} = \sum_{\beta\subseteq\alpha} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_r \frac{w_o G^+_\beta(x;r)}{G_\beta(a+\rho;r)}. \qquad \square$$

As mentioned earlier, the symmetric analogues of Theorems 1.1–1.3 have been established in [Ok]. In the case of symmetric Jack polynomials, expansions in the form of Corollary 1.9 were first considered by Bingham [B] ($r = 1/2$), and Lascoux [Lc] ($r = 1$), and, in general, by Lassalle [LS] and Kaneko [Ka]. The analogues of Theorems 1.6, 1.7, and Corollary 1.9 were obtained by Olshanski and Okounkov in [Ol], [OO1], and [OO2]; the analogues of Theorems 1.8 and 1.10 seem not to have been considered by them. Since these follow easily by our techniques, we formulate and prove them in Theorems 6.3 and 6.2.

While our proof follows the same general outline as Okounkov's argument, there are several differences. First, a decisive role is played by the affine Hecke algebra and Cherednik operators. The Hecke recursions satisfied by the G_α actually yield a simplification of part of the argument. On the other hand, there are some subtleties in the nonsymmetric case, as exhibited by the definition of G'_α.

Note. In the case of the Jack limit, the binomial coefficients have been recently (and independently) introduced in [BF]. The authors use Corollary 1.9 as the "definition" and deduce Corollary 6.6 as a consequence.

2. Preliminaries. We start by recalling certain basic properties of the $G_\alpha(x; q, t)$ (see [Kn] and [S3]).

The main result of [Kn, Theorem 3.6] is that the G_α satisfy the eigenequations

$$\Xi_i G_\alpha = \bar{\alpha}_i^{-1} G_\alpha$$

for the "inhomogeneous Cherednik operators" defined by

$$\Xi_i = x_i^{-1} + x_i^{-1} H_i \cdots H_{n-1} \Phi H_1 \cdots H_{i-1}.$$

In turn, the operators Φ and H_i are defined by

$$\Phi f(x_1, \ldots, x_n) = (x_n - t^{-n+1}) f(x_n/q, x_1, \ldots, x_{n-1}),$$

$$H_i = t s_i - (1-t) \frac{x_i}{x_i - x_{i+1}} (1 - s_i).$$

The H_i's satisfy the braid relations and the identity $(H_i - t)(H_i + 1) = 0$ and generate a representation of the Iwahori-Hecke algebra $\mathscr{H}$ of S_n on $\mathbb{F}[x]$.

Next, write $v^\# = (v_n - 1, v_1, \ldots, v_{n-1})$; let a be an indeterminate.

LEMMA 2.1. *We have*

(1) $\Phi f(a\bar{v}) = (a\bar{v}_n - t^{-n+1}) f(a\overline{v^\#})$;

(2) $H_i f(a\bar{v}) = ((t-1)\bar{v}_i)/(\bar{v}_i - \bar{v}_{i+1}) f(a\bar{v}) + (\bar{v}_i - t\bar{v}_{i+1})/(\bar{v}_i - \bar{v}_{i+1}) f(a\overline{s_i v})$.

This is proved just as in [Kn, Lemmas 2.1, 3.1]. The main point in (2) is that for $v \in \mathbb{Z}^n$, $s_i v = v \Rightarrow \bar{v}_i - t\bar{v}_{i+1} = 0$ and $s_i v \neq v \Rightarrow s_i \bar{v} = \overline{s_i v}$.

LEMMA 2.2. (1) *If* $\alpha_n > 0$, *then* $G_\alpha = q^{\alpha_n - 1}\Phi G_{\alpha^\#}$.
(2) *If* $\alpha_i > \alpha_{i+1}$, *then* $G_\alpha = (H_i + (1-t)d^{-1})G_{s_i\alpha}$ *where* $d = (1 - \bar{\alpha}_i/\bar{\alpha}_{i+1})$.

This is essentially in [Kn] and [S3]. Here is a sketch of the argument: Evidently the right sides of (1) and (2) have degree $\leqslant |\alpha|$, and by using Lemma 2.1, one verifies the vanishing conditions. It remains only to check that the coefficient of x^α is 1. This is obvious for (1), while for (2) one has to use the triangularity of Ξ_i (Lemma 3.10 of [Kn]).

In connection with Theorem 1.1, we define scalars $d_\alpha(q,t) = \prod_{s\in\alpha}(1 - q^{a(s)+1}t^{l(s)+1})$, $e_\alpha(q,t) = \prod_{s\in\alpha}(t^{1-n} - q^{a'(s)+1}t^{1-l'(s)})$, and $\phi_\alpha(a;q,t) = \prod_{s\in\alpha}(at^{l'(s)} - q^{a'(s)})$.

LEMMA 2.3. (1) *If* $\alpha_n > 0$, *then* $d_\alpha/d_{\alpha^\#} = 1 - t^n\bar{\alpha}_n$, $e_\alpha/e_{\alpha^\#} = t^{1-n} - t\bar{\alpha}_n$ *and* $\phi_\alpha(0) = -q^{\alpha_n-1}\phi_{\alpha^\#}(0)$.
(2) *If* $\alpha_i > \alpha_{i+1}$, *then* $d_\alpha = ((1 - \bar{\alpha}_i/\bar{\alpha}_{i+1})/(1 - t\bar{\alpha}_i/\bar{\alpha}_{i+1}))d_{s_i\alpha}$.
(3) $e_{w\alpha} = e_\alpha$ *and* $\phi_{w\alpha} = \phi_\alpha$ *for all* w *in* S_n.

The lemma can be proved in a manner very similar to Lemmas 4.1 and 4.2 in [S2]. To illustrate the argument, we sketch the proof of $e_\alpha/e_{\alpha^\#} = t^{1-n} - t\bar{\alpha}_n$; other proofs are similar. It follows from the definition of $\bar{\alpha}$ that $\bar{\alpha}_i = q^{\alpha_i}t^{-k_i}$, where $k_i = \#\{k < i \mid \alpha_k \geqslant \alpha_i\} + \#\{k > i \mid \alpha_k > \alpha_i\}$.

The diagram of α is obtained from $\alpha^\#$ by adding a point to the end of the first row and moving this row to the last place. The new point $s = (n, \alpha_n) \in \alpha$ has $a'(s) = \alpha_n - 1$ and $l'(s) = \#\{k < n \mid \alpha_k > \alpha_n\} = k_n$, while coarms and colegs of other points are unchanged. Thus $e_\alpha/e_{\alpha^\#} = t^{1-n} - q^{a'(s)+1}t^{1-l'(s)} = t^{1-n} - q^{\alpha_n}t^{1-k_n} = t^{1-n} - t\bar{\alpha}_n$.

We also need limit versions of these results, which are proved similarly. First, by [Kn, Theorem 6.6], we know that the $G_\alpha(x;r)$ satisfy the eigenequations

$$\tilde{\Xi}_i G_\alpha(x;r) = \bar{\alpha}(r)G_\alpha,$$

where the "limit" Cherednik operators are defined by

$$\tilde{\Xi}_i := x_i - \sigma_i \cdots \sigma_{n-1}\tilde{\Phi}\sigma_1 \cdots \sigma_{i-1}.$$

Here, $\sigma_i = s_i + r(x_i - x_{i+1})^{-1}(1 - s_i)$ is as in the previous section, and

$$\tilde{\Phi}f(x) = (x_n + (n-1)r)f(x_n - 1, x_1, \ldots, x_{n-1}).$$

LEMMA 2.4. *We have*
(1) $\tilde{\Phi}f(a + \bar{v}) = (a + \bar{v}_n + nr - r)f(a + \overline{v^\#})$;
(2) $\sigma_i f(a + \bar{v}) = (r/(\bar{v}_i - \bar{v}_{i+1}))f(a + \bar{v}) + ((\bar{v}_i - \bar{v}_{i+1} - r)/(\bar{v}_i - \bar{v}_{i+1}))f(a + \overline{s_i v})$.
□

LEMMA 2.5. (1) *If* $\alpha_n > 0$, *then* $G_\alpha = \tilde{\Phi} G_{\alpha^\#}$.
(2) *If* $\alpha_i > \alpha_{i+1}$, *then* $G_\alpha = (\sigma_i + rd^{-1})G_{s_i\alpha}$, *where* $d = \bar{\alpha}_i - \bar{\alpha}_{i+1}$. □

In connection with Theorem 1.6, we define scalars $d_\alpha(r) = \prod_{s\in\alpha}(a(s) + 1 + rl(s) + r)$, $e_\alpha(r) = \prod_{s\in\alpha}(a'(s) + 1 - rl'(s) + rn)$, and $\phi_\alpha(a; r) = \prod_{s\in\alpha}(a - a'(s) + rl'(s))$.

LEMMA 2.6. (1) *If* $\alpha_n > 0$, *then* $d_\alpha(r)/d_{\alpha^\#}(r) = rn + \bar{\alpha}_n(r) = e_\alpha(r)/e_{\alpha^\#}(r)$.
(2) *If* $\alpha_i > \alpha_{i+1}$, *then* $d_\alpha(r) = (d/(d+r))\, d_{s_i\alpha}(r)$, *where* $d = \bar{\alpha}_i - \bar{\alpha}_{i+1}$.
(3) *Let* $e_{w\alpha}(r) = e_\alpha(r)$ *and* $\phi_{w\alpha} = \phi_\alpha$ *for all* w *in* S_n. □

We now briefly discuss the symmetric case.

Definition. $R_\lambda(x; q, t)$ is the unique symmetric polynomial of degree $\leqslant |\lambda|$ that vanishes at $x = \bar{\mu}$ for partitions $\mu \neq \lambda$, $|\mu| \leqslant |\lambda|$; it is normalized so that the coefficient of x^λ is 1.

Definition. $R_\lambda(x; r)$ is the unique symmetric polynomial of degree $\leqslant |\lambda|$ that vanishes at $x = \bar{\mu}(r)$ for partitions $\mu \neq \lambda$, $|\mu| \leqslant |\lambda|$; it is normalized so that the coefficient of x^λ is 1.

The existence and uniqueness of $R_\lambda(x; q, t)$ was proved in [Kn] and [S2], as was the fact that its top term is the Macdonald polynomial $P_\lambda(q, t)$. In the case of $R_\lambda(x; r)$, these results were established in [S1] and [KS].

As in [S3, Theorem 4.6] and [Kn, Corollary 2.6], we have the following lemmas.

LEMMA 2.7. *Let* V_λ *be the* $\mathbb{F}$*-span of* $\{E_\alpha(x; q, t) | \alpha^+ = \lambda\}$. *Then* V_λ *is a module for the Hecke algebra* $\mathscr{H}$, *and* $V_\lambda^{\mathscr{H}} = \mathbb{F} R_\lambda(x; q, t)$.

LEMMA 2.8. *Let* $V_\lambda(r)$ *be the* $\mathbb{Q}(r)$*-span of* $\{E_\alpha(x; r) | \alpha^+ = \lambda\}$. *Then* $V_\lambda(r)$ *is a module for* $\sigma(S_n)$, *and* $V_\lambda(r)^{\sigma(S_n)} = \mathbb{Q}(r) R_\lambda(x; r)$.

Finally, for compatibility of notation between [Kn], [Ok], and [S3], we point out that

(1) [Kn] uses P_λ for R_λ, $\bar{P}_\lambda$ for P_λ, $\bar{E}_\alpha$ for E_α, and E_α for G_α;
(2) [Ok] uses $P^*_\lambda(x)$ for the "(shifted)" polynomial $R_\lambda(x\tau) \equiv R_\lambda(x_1, x_2 t^{-1}, \ldots, x_n t^{1-n})$ which vanishes at $(q_1^\mu, \ldots, q_n^\mu)$ and is symmetric in the variables $x_i t^{-i}$.
(3) [S3] uses $R_\lambda(x; q, t)$ to denote the polynomial $t^{-(n-1)|\lambda|} R_\lambda(x t^{n-1}; q^{-1}, t^{-1})$, which is symmetric and vanishes at the points $x = (q^{-\mu_1} t^{-n+1}, \ldots, q^{-\mu_{n-1}} t^{-1}, q^{-\mu_n})$; its top term is $P_\lambda(x; q^{-1}, t^{-1})$, which equals $P_\lambda(x; q, t)$ by [M1].

3. Evaluation. In this section, we prove the evaluation formulas, Theorems 1.1 and 1.6.

LEMMA 3.1. *For all* $w \in S_n$, *we have* $d_{w\alpha}(q, t) G_{w\alpha}(a\tau; q, t) = d_\alpha(q, t) G_\alpha(a\tau; q, t)$.

Proof. It suffices to verify this for $w = s_i$, and we may also assume that $\alpha_i > \alpha_{i+1}$.

Since $\tau = \overline{0}$, substituting $v = 0$ in Lemma 2.1 (2), we get $(H_i f)(a\tau) = tf(a\tau)$ for all functions f. Combining this with Lemma 2.2 (2), we get

$$G_\alpha(a\tau) = (t + (1-t)d^{-1})G_{s_i\alpha}(a\tau) = \frac{1 - t\bar{\alpha}_i/\bar{\alpha}_{i+1}}{1 - \bar{\alpha}_i/\bar{\alpha}_{i+1}}\, G_{s_i\alpha}(a\tau).$$

The result now follows from Lemma 2.3 (2). □

Theorem 1.1 states that $d_\alpha G_\alpha(a\tau) = e_\alpha \phi_\alpha(a\tau)$, and we first establish this for $a = 0$.

LEMMA 3.2. *We have*

$$d_\alpha G_\alpha(0) = e_\alpha \phi_\alpha(0) = e_\alpha \prod_{s\in\alpha}(-q^{a'(s)}).$$

Proof. The case $\alpha = 0$ is trivial, and we proceed by induction on $|\alpha|$ assuming $\alpha \neq 0$. By Lemma 3.1 and Lemma 2.3 (3), both sides are unchanged if we permute α, so we may assume that $\alpha_n > 0$ and that $d_{\alpha^\#} G_{\alpha^\#}(0) = e_{\alpha^\#}\phi_{\alpha^\#}(0)$. Thus it suffices to prove

$$\frac{G_\alpha(0)}{G_{\alpha^\#}(0)} = \left(\frac{e_\alpha}{e_{\alpha^\#}}\right)\left(\frac{d_{\alpha^\#}}{d_\alpha}\right)\left(\frac{\phi_\alpha(0)}{\phi_{\alpha^\#}(0)}\right).$$

The left-hand side can be computed by combining Lemmas 2.1 (1) and 2.2 (1), and the right-hand side can be computed by Lemma 2.3 (1). In each case, we get $-q^{\alpha_n - 1}t^{1-n}$. □

We now deduce Theorem 1.1 from the symmetric case (see [Ok]).

Proof of Theorem 1.1. If λ is a partition, then, by [Ok, formula (1.9)], $R_\lambda(a\tau)$ is an $\mathbb{F}$-multiple of $\phi_\lambda(a)$. Next, if α is a composition such that $\alpha^+ = \lambda$, then by Lemma 2.7, there are some coefficients $c_w \in \mathbb{F}$ such that $R_\lambda(x) = \sum_{w\in S_n} c_w d_{w\alpha} G_{w\alpha}(x)$. Evaluating at $x = a\tau$ and using Lemma 3.1, we get $R_\lambda(a\tau) = (\sum c_w)\, d_\alpha G_\alpha(a\tau)$. It follows that $d_\alpha(q,t)G_\alpha(a\tau)$ is an $\mathbb{F}$-multiple of $\phi_\lambda(a) = \phi_\alpha(a)$.

Setting $a = 0$ and using Lemma 3.2, we see that this multiple is $e_\alpha(q,t)$, and Theorem 1.1 follows. □

Proof of Theorem 1.6. Arguing as in Lemma 3.1, we deduce that $d_{w\alpha}(r)G_{w\alpha}(a+\rho;r) = d_\alpha(r)G_\alpha(a+\rho;r)$. Next, by [OO1] formula (2.3), $R_\lambda(a+\rho;r)$ is a $\mathbb{Q}(r)$-multiple of $\phi_\lambda(a;r)$. Arguing as before, we conclude that $d_\alpha(r)G_\alpha(a+\rho;r)$ is a $\mathbb{Q}(r)$-multiple of $\phi_\alpha(a;r)$.

Letting $a \to \infty$, we see that the multiple is $d_\alpha(r)E_\alpha(1;r)$, which equals $e_\alpha(r)$ by Theorem 1.3 of [S2]. The result follows. □

4. Reciprocity. In this section, we prove Theorems 1.2 and 1.7.

Proof of Theorem 1.2. Write $\mathbb{K} = \mathbb{F}(a) = \mathbb{Q}(q,t,a)$. For f in $\mathbb{K}[x]$, we have

$$\Xi_i f(a\bar{v}) = (a\bar{v}_i)^{-1} f(a\bar{v}) + (a\bar{v}_i)^{-1} H_i \cdots H_{n-1} \Phi H_1 \cdots H_{i-1} f(a\bar{v}).$$

Since $|v^{\#}| = |v| - 1$ and $|s_i v| = |v|$, it follows from Lemma 2.1 that the second term on the right-hand side is a combination of $f(a\bar{u})$ with $|u| = |v| - 1$, where the coefficients do not depend on f. Thus if p is a polynomial of degree d and we write $p(\Xi) \equiv p(\Xi_1, \ldots, \Xi_n)$, then $p(\Xi) f(a\tau) \equiv p(\Xi) f(a\bar{0}) = \sum_{|\beta| \leqslant d} c_p(\beta) f(a\tilde{\beta})$, with coefficients $c_p(\beta)$ independent of f.

Let $\mathscr{P}$ be the space of polynomials in $\mathbb{K}[x]$ of degree $\leqslant d$, and let $\mathscr{S}$ be the set of compositions β in $\mathbb{Z}_+^n$ with $|\beta| \leqslant d$. Then $p \mapsto c_p$ is a $\mathbb{K}$-linear map from $\mathscr{P}$ to $\mathbb{K}^{\mathscr{S}}$, and we claim that this map is bijective.

Since the spaces have the same dimension, it suffices to check injectivity. If $c_p = 0$, then $p(\Xi) f(a\tau) = 0$ for all f. In particular, setting $f = G_\beta$, we obtain $p(\bar{\beta}^{-1}) G_\beta(a\tau) = 0$. By Theorem 1.1, $G_\beta(a\tau) \neq 0$, and it follows that p vanishes at the points $\bar{\beta}^{-1} = \bar{\beta}(q^{-1}, t^{-1})$ for all β, and hence $p = 0$, proving injectivity.

Now fix α with $|\alpha| = d$, and let O_α be the polynomial in $\mathscr{P}$ whose image under $p \mapsto c_p$ is the delta function at α in $\mathbb{K}^{\mathscr{S}}$. Then O_α has degree $\leqslant |\alpha|$ and satisfies $O_\alpha(\Xi) f(a\tau) = f(a\tilde{\alpha})$ for all f. Setting $f = G_\beta$, we get $O_\alpha(\bar{\beta}^{-1}) G_\beta(a\tau) = G_\beta(a\tilde{\alpha})$. □

Proof of Theorem 1.7. This is proved similarly by using the limit Cherednik operators $\tilde{\Xi}_i$ and Lemma 2.4. □

5. The binomial formula. We now prove Theorems 1.3 and 1.8.

Proof of Theorem 1.3. Since the G'_β form a basis for $\mathbb{K}[x]$, there exist $b_{\beta\alpha} \in \mathbb{K}$ such that

$$(*) \qquad \frac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_{\beta : |\beta| \leqslant |\alpha|} b_{\beta\alpha} G'_\beta(x).$$

Substituting $x = \tilde{\gamma}$ and using Theorem 1.2, we get $O_\gamma(\bar{\alpha}^{-1}) = \sum_\beta b_{\beta\alpha} G'_\beta(\tilde{\gamma})$.

Let G be the (infinite) matrix whose entries are $g_{\gamma\beta} = G'_\beta(\tilde{\gamma})$. By Theorem 4.2 in [S3], polynomials of degree $\leqslant d$ are determined by their values at the points $\{\tilde{\gamma} : |\gamma| \leqslant d\}$. It follows that G has an inverse H, and we get $b_{\beta\alpha} = \sum_\gamma h_{\beta\gamma} O_\gamma(\bar{\alpha}^{-1})$. Since $G'_\beta(\tilde{\alpha}) = 0$ for $|\alpha| < |\beta|$, it follows that G and H are block triangular. Thus $h_{\beta\gamma} = 0$ for $|\gamma| > |\beta|$, and we deduce that $b_{\beta\alpha} = b_\beta(\bar{\alpha}^{-1})$, where $b_\beta := \sum_{\gamma : |\gamma| \leqslant |\beta|} h_{\beta\gamma} O_\gamma$ is a polynomial of degree $\leqslant |\beta|$.

The top-degree term on the left-hand side of $(*)$ is a multiple of E_α, and so by the definition of G'_α we obtain that $b_{\beta\alpha} = 0$ for $|\alpha| \leqslant |\beta|$, $\alpha \neq \beta$. Thus $b_\beta(\bar{\alpha}^{-1}) = 0$ for $|\alpha| \leqslant |\beta|$, $\alpha \neq \beta$, and since $\bar{\alpha}^{-1} = \bar{\alpha}(q^{-1}, t^{-1})$, it follows that $b_\beta(x)$ is a multiple of $G_\beta(x; q^{-1}, t^{-1})$. In other words, there are scalars c_β in $\mathbb{K}$ such that

$$\frac{G_\alpha(ax)}{G_\alpha(a\tau)} = \sum_\beta c_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{1/q, 1/t} G'_\beta(x).$$

Comparing the top-degree terms, we get $c_\alpha = a^{|\alpha|}/G_\alpha(a\tau)$, and the result follows. □

Proof of Theorem 1.8. The proof proceeds similarly using Theorem 1.7. □

6. More on the Jack limit. We now prove Theorem 1.10 and the symmetric versions of Theorems 1.8 and 1.10. Since the (q, t)-case is not considered in this section, we often suppress r to simplify the notation; for example, we write $G_\alpha(x)$ for $G_\alpha(x; r)$, $\tilde{\beta}$ for $\tilde{\beta}(r)$, and so on.

We start with a simple, but crucial, lemma.

LEMMA 6.1. *We have*
(1) $w_{-w_o\beta} = w_o w_\beta w_o$;
(2) $-w_o\tilde{\beta} = \tilde{\beta} + (n-1)r$.

Proof. For w in S_n, we have $(w_o w w_o)^{-1}(-w_o\beta) = (-w_o)(w^{-1}\beta)$, which is dominant if and only if $w^{-1}\beta$ is dominant. Since conjugation by w_o preserves length, part (1) follows.

Now $\tilde{\beta} = \overline{-w_o\beta} = -w_o\beta + w_{-w_o\beta}\rho = -w_o\beta + w_o w_\beta w_o\rho$ by part (1). Also, since $w_o\rho = -(n-1)r - \rho$, we get $\tilde{\beta} = -w_o(\beta + (n-1)r + w_\beta\rho) = -w_o(\bar{\beta} + (n-1)r)$. □

Proof of Theorem 1.10. For any polynomial f, wf and $\sigma(w)f$ have the same top terms. So, since $w_0^2 = 1$, the top term on the right-hand side of Theorem 1.10 is $(-1)^{|\alpha|} w_0^2 E_\alpha(-x) = E_\alpha(x)$. It remains only to show that the right-hand side of Theorem 1.10 belongs to the space V consisting of polynomials that vanish at the points $x = \tilde{\beta}$, $|\beta| < |\alpha|$.

Putting $a = 0$ and $v = \beta$ in Lemma 2.5 (2), we deduce that V is σ-invariant, and so it suffices to prove that $f \equiv w_o G_\alpha(-x - (n-1)r) \in V$. But, using Lemma 6.1, we get

$$f(\tilde{\beta}) = w_o G_\alpha(-\tilde{\beta} - (n-1)r) = G_\alpha(-w_o\tilde{\beta} - (n-1)r) = G_\alpha(\bar{\beta}),$$

which vanishes for $|\beta| < |\alpha|$ by the definition of G_α. □

We now turn to the symmetric versions of Theorems 1.8 and 1.10. As in [OO1], we define the "symmetric r-binomial coefficients" by

$$\binom{\lambda}{\mu}_r = \frac{R_\mu(\bar{\lambda}(r); r)}{R_\mu(\bar{\mu}(r); r)}.$$

The main result of [OO1] is the generalized binomial formula

$$(**)\qquad \frac{P_\lambda(1+x;r)}{P_\lambda(1;r)} = \sum_{\mu\subseteq\lambda} \binom{\lambda}{\mu}_r \frac{P_\mu(x;r)}{P_\mu(1;r)}.$$

For the inhomogeneous analogue of this result, we define the following.

Definition. $R'_\lambda(x;r)$ is the unique symmetric polynomial in $\mathbb{Q}(r)[x]$ such that
(1) $R'_\lambda(x;r)$ and $R_\lambda(x;r)$ have the same top-degree terms;
(2) $R'_\lambda(x;r)$ vanishes at $x = \tilde{\mu}(r) \equiv \overline{-w_o\mu}(r)$ for all μ with $|\mu| < |\lambda|$.

Then we have the following theorem.

6.2. THEOREM. *We have* $R'_\lambda(x;r) = (-1)^{|\lambda|} R_\lambda(-x-(n-1)r;r)$.

Proof. The two sides have the same top-degree terms, and it suffices to prove that the right-hand side vanishes for $x = \tilde{\mu}$ if $|\mu| < |\lambda|$. By symmetry, we may consider instead $x = w_o\tilde{\mu}$. Substituting this and using Lemma 6.1, the right-hand side becomes $(-1)^{|\lambda|}R_\lambda(\bar{\mu};r)$, which vanishes by the definition of R_λ. □

THEOREM 6.3. *We have*

$$\frac{R_\lambda(a+x;r)}{R_\lambda(a+\rho;r)} = \sum_{\mu\subseteq\lambda} \binom{\lambda}{\mu}_r \frac{R'_\mu(x;r)}{R_\mu(a+\rho;r)}.$$

We deduce Theorem 6.3 from Theorem 1.8 by symmetrization. Write $\mathscr{S}$ for the operator

$$\frac{1}{n!}\sum_{w\in S_n} \sigma(w)$$

acting on $\mathbb{Q}(r)[x]$.

LEMMA 6.4. *$\mathscr{S}$ maps polynomials to symmetric polynomials.*

Proof. For all i, we have $\sigma_i\mathscr{S} = \sum_{w\in S_n}\sigma(s_iw) = \mathscr{S}$. So if f is a polynomial in the image of $\mathscr{S}$, then $(1-\sigma_i)f = 0$. Rewriting this, we get

$$\left(1 - \frac{r}{x_i - x_{i+1}}\right)(1-s_i)f = 0.$$

Hence $(1-s_i)f = 0$ for all i, which implies that f is symmetric. □

LEMMA 6.5. *Let α be any composition with $\alpha^+ = \lambda$; then*
(1) $(\mathscr{S}G_\alpha(a+x))/(G_\alpha(a+\rho)) = (R_\lambda(a+x))/(R_\lambda(a+\rho))$;
(2) $(\mathscr{S}G'_\alpha(x))/(G_\alpha(a+\rho)) = (R'_\lambda(x))/(R_\lambda(a+\rho))$.

Proof. If $|\beta| \leqslant |\alpha|$ and $\beta^+ \neq \alpha^+$, then Lemma 2.4 implies that, for all w in S_n, the polynomial $\sigma(w)G_\alpha(x)$ vanishes at $x = \bar{\beta}$. This means that $f = \mathscr{S}G_\alpha(a+x)$ vanishes at $\bar{\mu} - a$ for all partitions μ satisfying $|\mu| \leqslant |\lambda|$, $\mu \neq \lambda$. Since f is symmetric and of the right degree, we conclude that f is a multiple of $R_\lambda(a+x)$. To determine the multiple, we merely evaluate both sides of (1) at $x = \rho$ and use the fact that $\sigma(w)G_\alpha(a+\rho) = G_\alpha(a+\rho)$ (which follows from Lemma 2.4 (2)). This proves (1).

For (2), the same argument proves that $\mathscr{S}G'_\alpha(x)$ vanishes at $\bar{\mu}$ for $|\mu| < |\lambda|$. To finish the proof, it suffices to prove that the *top* terms of the two sides are equal. But these are also the top terms of (1) and hence are equal. □

Proof of Theorem 6.3. Fix α with $\alpha^+ = \lambda$, and apply $\mathscr{S}$ to both sides of Theorem 1.8. By Lemma 6.5, we get

$$\frac{R_\lambda(a+x;r)}{R_\lambda(a+\rho;r)} = \sum_{\mu \subseteq \lambda} k_\mu \frac{R'_\mu(x;r)}{R_\mu(a+\rho;r)} \quad \text{with } k_\mu = \sum_{\beta^+=\mu} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_r \in \mathbb{Q}(r).$$

To conclude, we need to establish that $k_\mu = \binom{\lambda}{\mu}_r$, but this follows by putting $x = ax$ in the above, letting $a \to \infty$, and using $(**)$. □

COROLLARY 6.6. *For each α satisfying $\alpha^+ = \lambda$, we have*

$$\sum_{\beta^+=\mu} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_r = \binom{\lambda}{\mu}_r. \qquad \square$$

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