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A recursion and a combinatorial formula for Jack polynomials

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1. Introduction

The Jack polynomials $J_\lambda(x;\alpha)$ form a remarkable class of symmetric polynomials. They are indexed by a partition λ and depend on a parameter α. One of their properties is that several classical families of symmetric functions can be obtained by specializing α, e.g., the monomial symmetric functions m_λ ($\alpha=\infty$), the elementary functions $e_{\lambda'}$ ($\alpha=0$), the Schur functions s_λ ($\alpha=1$) and finally the two classes of zonal polynomials ($\alpha=2$, $\alpha=1/2$).

The Jack polynomials can be defined in various ways, e.g.:

a) as an orthogonal family of functions which is compatible with the canonical filtration of the ring of symmetric functions or
b) as simultaneous eigenfunctions of certain differential operators (the Sekiguchi–Debiard operators).

Recently Opdam, [O], constructed a similar family $F_\lambda(x;\alpha)$ of *non-symmetric* polynomials. The index runs now through all compositions $\lambda\in\mathbb{N}^n$. They are defined in a completely similar fashion, e.g., the Sekiguchi–Debiard operators are being replaced by the Cherednik differential-reflection operators (see Sect. 3). It is becoming more and more clear that these polynomials are as important as their symmetric counterparts.

The purpose of this paper is to add to the existing characterizations of Jack polynomials two further ones:

c) a recursion formula among the F_λ together with two formulas to obtain J_λ from them.
d) combinatorial formulas of both J_λ and F_λ in terms of certain generalized tableaux.

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There are many advantages of these new characterizations over the ones mentioned above. In a) and b), the existence of functions with these properties is not obvious and requires a proof whereas c) and d) could immediately serve as a *definition* of Jack polynomials. Moreover, a) and b) determine the functions only up to a scalar while c) and d) give automatically the right normalization.

More importantly, our formulas are explicit enough such that both the recursion relation and the combinatorial formula enable us to prove a conjecture of Macdonald ([M, S]). For a partition λ let $m_i(\lambda)$ be the number of parts which are equal to i and let $u_\lambda := \prod_{i \geqq 1} m_i(\lambda)!$. Then we prove

1.1. Theorem. *Let* $J_\lambda(x;\alpha) = \sum_\mu v_{\lambda\mu}(\alpha) m_\mu(x)$. *Then all functions* $\tilde{v}_{\lambda\mu}(\alpha) := u_\mu^{-1} v_{\lambda\mu}(\alpha)$ *are polynomials in* α *with positive integral coefficients.*

For an analogous statement for the F_λ see Theorem 4.11. We would like to mention the recent papers [LV1] and [LV2] of Lapointe and Vinet which, by completely different methods, establish that $v_{\lambda\mu}$ is a polynomial with integral coefficients. Except for special cases, before that it was not even known that $v_{\lambda\mu}$ is a polynomial.

We continue with the description of c) and d). First, the recursion formula.

For $\lambda \in \mathbb{N}^n$ we define the *degree* $|\lambda| := \sum_i \lambda_i$. Its *length* $l(\lambda)$ is the maximal index i such that $\lambda_i \neq 0$. With $m := l(\lambda)$ we define $\tilde{\lambda}_m := \alpha\lambda_m + k + 1$ where k is the number of indices $i = 1, \ldots, m-1$ with $\lambda_i < \lambda_m$. Moreover, let $\lambda^* := (\lambda_m - 1, \lambda_1, \ldots, \lambda_{m-1}, 0, \ldots, 0)$. For $i = m, \ldots, n$ let

$$f_i(x) := F_{\lambda^*}(x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\ .$$

Then we prove (Theorem 4.6):

$$F_\lambda(x) = \tilde{\lambda}_m x_m f_m(x) + x_{m+1} f_{m+1}(x) + x_{m+2} f_{m+2}(x) + \cdots + x_n f_n(x)\ .$$

The symmetric functions are most easily obtained if the number of variables is big enough, i.e., $n \geqq 2m$. Let $\lambda^+ \in \mathbb{N}^{n-m}$ be the partition which is a permutation of $(\lambda_1, \ldots, \lambda_{n-m})$. Then we prove (Theorem 4.10)

$$J_{\lambda^+}(z_{m+1}, \ldots, z_n) = F_\lambda(0, \ldots, 0, z_{m+1}, \ldots, z_n)\ .$$

Now, we describe the combinatorial formula. For simplicity we restrict ourselves to the symmetric case J_λ. Let λ be a partition. A *generalized tableau of shape* λ is a labelling T of the boxes in the Ferrers diagram of λ by numbers $1, 2, \ldots, n$. To T, we associate the monomial $x^T := \prod_{s \in \lambda} x_{T(s)}$.

We call T *admissible* if it satisfies for all boxes $(i,j) \in \lambda$:

a) $T(i,j) \neq T(i',j)$ whenever $i' > i$

b) $T(i,j) \neq T(i',j-1)$ whenever $j > 1$ and $i' < i$.

A box $s = (i,j) \in \lambda$ is *critical* (for T) if $j > 1$ and $T(i,j) = T(i,j-1)$.

Let λ' be the dual partition to λ. The armlength of $s = (i,j) \in \lambda$ is defined as $a_\lambda(s) := \lambda_i - j$. Likewise, the leglength is defined as $l_\lambda(s) := \lambda'_j - i$. Then we introduce the linear polynomial $d_\lambda(s) := \alpha(a_\lambda(s)+1)+(l_\lambda(s)+1)$. With $d_T(\alpha) := \prod_{s\,\text{critical}} d_\lambda(s)$ our formula reads (Theorem 5.1)

$$J_\lambda(x;\alpha) = \sum_{T\ \text{admissible}} d_T(\alpha)x^T\ .$$

This formula immediately implies the Macdonald conjecture. Consider a partition μ and the set $\mathscr{T}$ of all tableaux T with $x^T = x^\mu$. Let H be the group of permutations π of the labels $1,\dots,n$ such that $\mu_{\pi(i)} = \mu_i$ for all i and $\pi(i) = i$ whenever $\mu_i = 0$. This group acts freely on $\mathscr{T}$ by permuting the labels such that $d_T(\alpha)$ and x^T are left invariant. Since the order of H is u_μ, we obtain that the coefficient of x^μ is divisible by u_μ.

In the sequel we prove first that the eigenfunctions of the Cherednik operators satisfy our recursion formula. Then we prove that the functions defined by the combinatorial formula satisfy the recursion relation as well.

2. The definition of Jack polynomials

Most constructions and results in the following two sections can be found in Opdam's paper [O] in the framework of arbitrary root systems. Here we are only interested in the case of A_{n-1}.

Let $\mathscr{P} := \mathbb{Q}[x_1,\dots,x_n]$ be the ring of polynomials. For an indeterminate α let $\mathscr{P}_\alpha = \mathscr{P}\otimes_{\mathbb{Q}}\mathbb{Q}(\alpha)$. If α is such that $1/\alpha$ is a non-negative *integer* then $\delta^{1/\alpha}(x) := \prod_{i\neq j}(1-x_ix_j^{-1})^{1/\alpha}$ is in the Laurent polynomial ring $\mathscr{P}' = \mathscr{P}[x^{-1}]$. Let $[f]_0 \in \mathbb{Q}$ denote the constant term of $f\in\mathscr{P}'$. Then

$$\langle f,g\rangle_\alpha := [f(x)g(x^{-1})\delta^{1/\alpha}(x)]_0$$

defines a non-degenerate scalar product on $\mathscr{P}$.

Consider $\Lambda := \mathbb{N}^n$. The *degree* of $\lambda = (\lambda_i)\in\Lambda$ is defined as $|\lambda| := \sum_i \lambda_i$ and its *length* as $l(\lambda) := \max\{k \mid \lambda_k \neq 0\}$ (with $l(0) := 0$). We recall the (partial) order relation of [O] on Λ. We start with the usual ordering on the set $\Lambda^+\subseteq\Lambda$ of all partitions $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$. Here $\lambda\geqq\mu$ if $|\lambda| = |\mu|$ and

$$\lambda_1+\lambda_2+\cdots+\lambda_i \geqq \mu_1+\mu_2+\cdots+\mu_i \quad \text{for all } i = 1,\dots,n\ .$$

This order relation is extended to all of Λ as follows. Clearly, the symmetric group W on n letters acts on Λ and for every $\lambda\in\Lambda$ there is a unique partition λ^+ in the orbit $W\lambda$. For all permutations $w\in W$ with $\lambda = w\lambda^+$ there is a unique one, denoted by w_λ, of minimal length. We define $\lambda\geqq\mu$ if either $\lambda^+ > \mu^+$ or $\lambda^+ = \mu^+$ and $w_\lambda \leqq w_\mu$ in the Bruhat order of W. In particular, λ^+ is the unique *maximum* of $W\lambda$.

Non-symmetric Jack polynomials are defined by the following theorem. Here x^λ be the monomial corresponding to λ.

2.1. Theorem. ([O] 2.6) *For every $\lambda \in \Lambda$ there is a unique polynomial $E_\lambda(x;\alpha) \in \mathscr{P}_\alpha$ satisfying*
i) $E_\lambda = x^\lambda + \sum_{\mu\in\Lambda:\mu<\lambda} c_{\lambda\mu}(\alpha)x^\mu$ *and*
ii) $\langle E_\lambda, x^\mu\rangle_\alpha = 0$ *for all $\mu \in \Lambda$ with $\mu < \lambda$ and almost all α such that $1/\alpha \in \mathbb{N}$.*
Moreover, the collection $\{E_\lambda \mid \lambda \in \Lambda\}$ forms a $\mathbb{Q}(\alpha)$-linear basis of $\mathscr{P}_\alpha$.

The symmetric group W acts on $\mathscr{P}$ in the obvious way. Then $\mathscr{P}^W$ is the algebra of symmetric functions. For $\lambda \in \Lambda^+$ let $m_\lambda := \sum Wx^\lambda$ denote the corresponding monomial symmetric function. Then (symmetric) Jack polynomials are defined by:

2.2. Theorem. ([M] 10.13) *For every $\lambda \in \Lambda^+$ there is a unique symmetric polynomial $P_\lambda(x;\alpha) \in \mathscr{P}_\alpha^W$ satisfying*
i) $P_\lambda = m_\lambda + \sum_{\mu\in\Lambda^+:\mu<\lambda} c'_{\lambda\mu}(\alpha)m_\mu$ *and*
ii) $\langle P_\lambda, m_\mu\rangle_\alpha = 0$ *for all $\mu \in \Lambda^+$ with $\mu < \lambda$ and almost all α with $1/\alpha \in \mathbb{N}$.*
Moreover, the collection $\{P_\lambda \mid \lambda \in \Lambda\}$ forms a $\mathbb{Q}(\alpha)$-linear basis of $\mathscr{P}_\alpha^W$.

An easy consequence of the definitions is:

2.3. Lemma. *For $\lambda \in \Lambda^+$ let $\mathscr{P}_\lambda \subset \mathscr{P}_\alpha$ be the $\mathbb{Q}(\alpha)$-linear subspace spanned by the $E_{w\lambda}$, $w \in W$. Then $\mathscr{P}_\lambda$ is W-stable and $\mathscr{P}_\lambda^W = \mathbb{Q}(\alpha)P_\lambda$.*

The action of $w \in W$ on $\mathscr{P}_\lambda$ is, in general, difficult to describe in terms of the basis $E_{w\lambda}$, but, for a simple reflection $s_i := (i, i+1) \in W$, this is possible. We first present only a special case and the rest later (Proposition 4.3).

2.4. Lemma. *Let $\lambda \in \Lambda$ with $\lambda_i = \lambda_{i+1}$. Then $s_iE_\lambda = E_\lambda$.*

Proof. This follows directly from the definition and the fact that $\mu < \lambda = s_i\lambda$ implies $s_i\mu < \lambda$. □

One consequence of this lemma is that if $\lambda_i = 0$ for all $i > m$ then E_λ is symmetric in the variables $x_{m+1}, \ldots, x_n$. This fact will be crucial later on.

3. Definition of Cherednik's operators

As already mentioned, the symmetric group W acts on $\mathscr{P}$. For $i \neq j$ let $s_{ij} \in W$ denote the transposition (ij). Then

$$N_{ij} := \frac{1 - s_{ij}}{x_i - x_j}$$

is a well defined operator on $\mathscr{P}$. Next, for $i = 1, \ldots, n$ we define the following differential-reflection operators, which were first studied by Cherednik [C] (see also [O]):

$$\xi_i := \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{i-1} N_{ij}x_j + \sum_{j=i+1}^{n} x_jN_{ij}$$

Remark. The operators in [O] depend on the choice of a positive root system. We use $\{-x_1+x_2,\ldots,-x_{n-1}+x_n\}$ as the set of simple roots. This has the advantage that the ξ_i are stable under adding variables.

The ξ_i commute pairwise. This is most easily seen by using Corollary 3.2 below. Furthermore, they satisfy the following commutation relations with the simple reflections $s_i = s_{i\,i+1}$. This one checks by direct calculation.

$$\xi_i s_i - s_i \xi_{i+1} = 1$$

$$\xi_{i+1} s_i - s_i \xi_i = -1$$

$$\xi_i s_j - s_j \xi_i = 0 \quad j \neq i, i+1$$

(In other words, the s_j and ξ_i generate a graded Hecke algebra.)

3.1. Lemma. (a) *The action of ξ_i on $\mathcal{P}$ is triangular. More precisely*

$$\xi_i(x^\lambda) = \bar{\lambda}_i x^\lambda + \sum_{\mu \in \Lambda : \mu < \lambda} c_\mu x^\mu$$

where $\bar{\lambda}_i := \alpha\lambda_i - (k_i' + k_i'')$ *with*

$$k_i' = \#\{j = 1,\ldots,i-1 \mid \lambda_j \geqq \lambda_i\}$$

$$k_i'' = \#\{j = i+1,\ldots,n \mid \lambda_j > \lambda_i\}$$

(b) *For* $1/\alpha \in \mathbb{N}$, *the operator ξ_i is symmetric with respect to the scalar product* $\langle , \rangle_\alpha$.

Proof. (a) is [O] 2.10 and (b) is [C] 3.8. The key to part (a) is the observation that $(N_{ij}x_j)(x_i^a x_j^b)$ contains $x_i^a x_j^b$ if and only if $a \leqq b$ while for $(x_j N_{ij})(x_i^a x_j^b)$ one needs $a < b$. □

3.2. Corollary. ([O] 2.7) *The E_λ form a simultaneous eigenbasis for the ξ_i. More precisely*, $\xi_i(E_\lambda) = \bar{\lambda}_i E_\lambda$.

Remarks. 1. For an alternate proof for the existence of a simultaneous eigenbasis see the remark after Theorem 4.6 below.
2. The eigenvalues $\bar{\lambda}_i$ could be more concisely described as follows. Consider the vector $\varrho := (0,-1,-2,\ldots,-n+1)$. Then $\bar{\lambda}_i = (\alpha\lambda + w_\lambda \varrho)_i$.

Another consequence is stability:

3.3. Corollary. *Let $\lambda \in \Lambda$ with $\lambda_n = 0$ and $\lambda' := (\lambda_1,\ldots,\lambda_{n-1})$. Then we have*

$$E_\lambda|_{x_n=0} = E_{\lambda'} \in \mathbb{Q}(\alpha)[x_1,\ldots,x_{n-1}]\,.$$

If λ is a partition, then

$$P_\lambda|_{x_n=0} = P_{\lambda'} \in \mathbb{Q}(\alpha)[x_1,\ldots,x_{n-1}]\,.$$

Proof. Obviously, when substituting $x_n = 0$, the operators $\xi_1, \ldots, \xi_{n-1}$ induce their counterpart on $\mathbb{Q}(\alpha)[x_1, \ldots, x_{n-1}]$. Hence, the first statement follows from Corollary 3.2 and then the second from Lemma 2.3. □

Remark. This Corollary allows to define E_λ and P_λ in infinitely many variables $x_1, x_2, x_3, \ldots$ where $\lambda \in \mathbb{N}^\infty$ is a sequence such that almost all λ_i are zero. More precisely, they lie in $\mathscr{P}^\infty := \varprojlim \mathbb{Q}(\alpha)[x_1, \ldots, x_n]$ where the limit is to be taken in the category of graded algebras. Actually, Lemma 2.4 implies that the E_λ even lie in the subalgebra $\mathscr{P}^{(\infty)}$ of *almost symmetric* functions, i.e., those $f \in \mathscr{P}^\infty$ which are symmetric in the variables $x_m, x_{m+1}, \ldots$ for some $m \geqq 1$ depending on f.

4. The recursion formula

We define "creation operators" for the E_λ. The first one is very easy to define but seems to be new:

$$\Phi := x_n s_{n-1} s_{n-2} \cdots s_1 \ ,$$

i.e.,

$$(\Phi f)(x_1, \ldots, x_n) := x_n f(x_n, x_1, \ldots, x_{n-1}) \quad (f \in \mathscr{P})\ .$$

4.1. Lemma. *The following relations hold*:

$$\xi_i \Phi = \Phi \xi_{i+1} \quad \textit{for } i = 1, \ldots, n-1$$

$$\xi_n \Phi = \Phi(\xi_1 + 1)$$

Proof. Let $\tau = s_{n-1} \cdots s_1$. This is a cyclic permutation with $x_n \tau = \tau x_1$. Then the assertion follows from the following commutation relations which hold for all $1 \leqq i \neq j < n$:

$$x_i \partial_{x_i} \Phi = \Phi x_{i+1} \partial_{x_{i+1}}, \quad x_n \partial_{x_n} \Phi = \Phi x_1 \partial_{x_1} + \Phi\ .$$

$$N_{ij} x_j \Phi = \Phi N_{i+1\, j+1} x_{j+1}, \quad x_j N_{ij} \Phi = \Phi x_{j+1} N_{i+1\, j+1}$$

$$x_n N_{in} \Phi = x_n N_{in} x_n \tau = x_n \tau N_{i+1\, 1} x_1 = \Phi N_{i+1\, 1} x_1$$

$$N_{nj} x_j \Phi = N_{nj} x_n x_j \tau = x_n x_j N_{nj} \tau = \Phi x_{j+1} N_{1\, j+1}$$ □

4.2. Corollary. *Let $\lambda \in \Lambda$ with $\lambda_n \neq 0$. Put $\lambda^* := (\lambda_n - 1, \lambda_1, \ldots, \lambda_{n-1})$. Then $E_\lambda = \Phi(E_{\lambda^*})$.*

Opdam [O] 1.2 constructed an operator which permutes two entries:

4.3. Proposition. *Let $i \in \{1, \ldots, n-1\}$ and $\lambda \in \Lambda$ with $\lambda_i > \lambda_{i+1}$. Then $xE_\lambda = (xs_i + 1)E_{s_i(\lambda)}$ with $x = \bar{\lambda}_i - \bar{\lambda}_{i+1}$.*

Proof. Let $E := (xs_i + 1)E_{s_i(\lambda)}$. Then one easily verifies $\xi_j(E) = \bar{\lambda}_j E$ for all j. The assertion follows by comparing the highest coefficient. □

These operators together with Φ already suffice to generate all E_λ, but we still have to divide by the factor x. We prove a refinement.

4.4. Lemma. *For* $\lambda \in \Lambda$ *with* $1 \leqq m := l(\lambda) \leqq n$ *let* $\lambda^\sharp := (\lambda_1, \ldots, \lambda_{m-1}, 0, \ldots, 0, \lambda_m)$*. Then* $(\bar{\lambda}_m + m)E_\lambda = X_\lambda(E_{\lambda^\sharp})$ *where*

$$X_\lambda := (\bar{\lambda}_m + m)s_m \cdots s_{n-1} + \sum_{i=m+1}^{n} s_i s_{i+1} \cdots s_{n-1}$$

Proof. We prove the statement by induction on $n - m$, the number of trailing zeros. If $m = n$ then X_λ is just multiplication by $(\bar{\lambda}_n + n)$. For $m = n - 1$, the assertion follows from Proposition 4.3. Assume now $m \leqq n - 2$ and put $\lambda^\circ := (\lambda_1, \ldots, \lambda_{m-1}, 0, \lambda_m, 0, \ldots, 0)$. It follows from Lemma 3.1 that $\bar{\lambda}_{m+1} = -m$ and $\bar{\lambda}^\sharp_{m+1} = \bar{\lambda}_m$. Put $x := \bar{\lambda}_m + m$, $\zeta_i = s_i \cdots s_{n-1}$, and $\tau_j := \sum_{i=j+1}^{n} \zeta_i$. Then, by induction and Proposition 4.3, we get $x(x+1)E_\lambda = (xs_m + 1)[(x+1)\zeta_{m+1} + \tau_{m+1}]E_{\lambda^\sharp} = [x(x+1)\zeta_m + xs_m\tau_{m+1} + (x+1)\zeta_{m+1} + \tau_{m+1}]E_{\lambda^\sharp}$. Now we use that s_m commutes with τ_{m+1} and that $s_m E_{\lambda^\sharp} = E_{\lambda^\sharp}$ (Lemma 2.4). Thus we obtain $x(x+1)E_\lambda = (x+1)[x\zeta_m + \tau_{m+1} + \zeta_{m+1}]E_{\lambda^\sharp} = (x+1)X_\lambda E_{\lambda^\sharp}$. Finally, $x + 1 \neq 0$ since $\lambda_m \neq 0$. □

Now, we introduce another normalization of the Jack polynomials. Recall that the *diagram* of $\lambda \in \Lambda$ is the set of points (or *boxes*) $(i, j) \in \mathbb{Z}^2$ such that $1 \leqq i \leqq n$ and $1 \leqq j \leqq \lambda_i$. As usual, we identify λ with its diagram. For each box $s = (i, j) \in \lambda$ we define the *arm-length* $a_\lambda(s)$, the *leg-length* $l_\lambda(s)$ and the α-*hooklengths* $c_\lambda(s)$, $d_\lambda(s)$ as follows:

$$\begin{aligned}
a_\lambda(s) &:= \lambda_i - j \\
l'_\lambda(s) &:= \#\{k = 1, \ldots, i-1 \mid j \leqq \lambda_k + 1 \leqq \lambda_i\} \\
l''_\lambda(s) &:= \#\{k = i+1, \ldots, n \mid j \leqq \lambda_k \leqq \lambda_i\} \\
l_\lambda(s) &:= l'_\lambda(s) + l''_\lambda(s) \\
c_\lambda(s) &:= \alpha a_\lambda(s) + (l_\lambda(s) + 1) \\
d_\lambda(s) &:= \alpha(a_\lambda(s) + 1) + (l_\lambda(s) + 1)
\end{aligned}$$

Now, we define

$$F_\lambda(x; \alpha) := \prod_{s \in \lambda} d_\lambda(s) E_\lambda(x; \alpha) ;$$

$$J_\lambda(x; \alpha) := \prod_{s \in \lambda} c_\lambda(s) P_\lambda(x; \alpha) .$$

If $\lambda \in \Lambda^+$ is a partition then $l'(s) = 0$ and $l''(s) = l(s)$ is just the usual leg-length. Moreover, $c_\lambda(s)$ is called the lower hook length in [S]. This also shows that our $J_\lambda(x; \alpha)$ coincides with $J_\lambda^{(\alpha)}$ in [M].

First we state a simple lemma which calculates $d_\lambda(s)$ in a special case.

4.5. Lemma. *Let* $\lambda \in \Lambda$ *and* $s = (i, 1) \in \lambda$. *Then* $d_\lambda(s) = \bar{\lambda}_i + i + a_0$ *where* $a_0 := \#\{k = i+1, \ldots, n \mid \lambda_k > 0\}$.

Proof. Follows directly from the definitions. □

Now we can prove our main recursion formula:

4.6. Theorem. *For any* $1 \leqq k \leqq n$ *put*

$$\Phi_k := x_k s_{k-1} \cdots s_1$$

For $\lambda \in \Lambda$ *with* $m := l(\lambda) > 0$ *let*

$$\lambda^* := (\lambda_m - 1, \lambda_1, \ldots, \lambda_{m-1}, 0, \ldots, 0)\ ;$$

$$Y_\lambda := X_\lambda \Phi = (\bar{\lambda}_m + m)\Phi_m + \Phi_{m+1} + \cdots + \Phi_n$$

Then $F_\lambda = Y_\lambda(F_{\lambda^*})$.

Proof. Corollary 4.2 and Lemma 4.4 imply $xE_\lambda = Y_\lambda(E_{\lambda^*})$ with $x = \bar{\lambda}_m + m$. The diagram of λ^* is obtained from λ by taking the last non-empty row, removing its first box $s_0 = (m, 1)$ and putting the rest on top. One easily checks from the definitions that the arm-length and the leg-length of the remaining boxes do not change. Moreover $x = d_\lambda(s_0)$ by Lemma 4.5. This proves the theorem. □

Remark. One could use Theorem 4.6 as a *definition* of F_λ. Then reading the proofs of Lemma 4.4 and Theorem 4.6 backwards one sees that the so defined functions are simultaneous eigenfunctions for the Cherednik operators. This gives an alternate proof of Corollary 3.2 and of the commutativity of Cherednik operators.

The following Corollary shows another way to normalize non-symmetric Jack polynomials in the case the number of variables is large enough. It is an analogue of Stanley's normalization in [S].

4.7. Corollary. *Let* $\lambda \in \Lambda$, *put* $d := |\lambda|$ *and* $m := l(\lambda)$. *Assume* $n \geqq m + d$. *Then the coefficient of* $x_{m+1} \cdots x_{m+d}$ *in* F_λ *is* $d!$.

Proof. We have

$$c := x_{m+1} \cdots x_{m+d} = \Phi_i(x_{m+2}, \ldots, x_{m+d})$$

for $i = m+1, \ldots, m+d$ and this is the only way, c can arise as the image of an operator Φ_i. Hence, Theorem 4.6 implies that the coefficient of c in F_λ is d times the coefficient of $x_{m+2} \cdots x_{m+d}$ in F_{λ^*}. But F_{λ^*} is symmetric in the variables $x_{m^*+1}, \ldots, x_n$ where $m^* = l(\lambda^*)$. The assertion follows by induction. □

We give the first of two ways how to obtain the symmetric Jack polynomials from the non-symmetric ones. Before we do so, recall some notation. For any $\lambda \in \Lambda$ let $m_i(\lambda) := \#\{k \mid \lambda_k = i\}$ and $u_\lambda := \prod_{i \geqq 1} m_i(\lambda)!$.

4.8. Theorem. *For $\lambda \in \Lambda^+$ with $m := l(\lambda)$ put*

$$\lambda^0 := (\lambda_m - 1, \ldots, \lambda_1 - 1, 0, \ldots, 0) .$$

Then $J_\lambda(x;\alpha) = \frac{1}{(n-m)!} \sum_{w \in W} w\Phi^m(F_{\lambda^0})$.

Proof. Denote the right hand side by J. Let $\lambda^- := (0, \ldots, 0, \lambda_m, \ldots, \lambda_1)$. Then Corollary 4.2 implies that $F' := \Phi^m(F_{\lambda^0})$ is proportional to F_{λ^-}. Lemma 2.3 implies that J and J_λ are proportional.

To see that they are equal, it suffices to compare the coefficients of x^{λ^-}. Since Φ does not change the leading coefficient, the coefficient of z^{λ^-} in F' is $\prod_{\substack{s\in\lambda^- \\ s \neq (i,1)}} d_{\lambda^-}(s)$. Since λ^- is minimal in $W\lambda$, no other monomial occurring in F' is conjugated to x^{λ^-}. Moreover, F' is invariant for the isotropy group W_{λ^-}. Its order is $(n-m)!u_\lambda$. Hence the coefficient of x^{λ^-} in J is

$$u_\lambda \prod_{\substack{s\in\lambda^- \\ s \neq (i,1)}} d_{\lambda^-}(s)$$

On the other hand, by definition, the coefficient of x^{λ^-} in J_λ is

$$\prod_{s\in\lambda} c_\lambda(s) .$$

Let $w \in W$ be the shortest permutation with $w(\lambda) = \lambda^-$. This means $w(i) > w(j)$ whenever $\lambda_i > \lambda_j$ but $w(i) < w(j)$ for $\lambda_i = \lambda_j$ and $i < j$. Consider the following correspondence between boxes of λ and λ^-:

$$\lambda \ni s = (i,j) \leftrightarrow s^- = (\pi(i), j+1) \in \lambda^- .$$

This is defined for all s with $j < \lambda_i$. One easily verifies that $a_\lambda(s) = a_{\lambda^-}(s^-) + 1$ and $l_\lambda(s) = l_{\lambda^-}(s^-)$. Hence, $c_\lambda(s) = d_{\lambda^-}(s^-)$, i.e., s and s^- contribute the same factor to the products above. What is left out of the correspondence are those boxes of λ with $j = \lambda_i$ and the first column of λ^-. The first type of these boxes contributes u_λ to the factor of J_λ. The second type doesn't contribute by construction. This shows that $J_\lambda = J$. □

This proof gives a bit more, namely a result of Stanley ([S] Theorem 1.1 in conjunction with Theorem 5.6).

4.9. Corollary. *Let $\lambda \in \Lambda^+$ with $d := |\lambda| \leqq n$. Then the coefficient of m_{1^d} in J_λ is $d!$.*

Proof. We keep the notation of the proof of Theorem 4.8. We have

$$F' = \Phi^m(F_{\lambda^0}) = x_{n-m+1} \cdots x_n F_{\lambda_0}(x_{n-m+1}, \ldots, x_n, x_1, \ldots, x_{n-m}) .$$

Hence every monomial occurring in F' which contains each variable with a power of at most one is of the form $x_{i_1} \cdots x_{i_{d-m}} x_{n-m+1} \cdots x_n$ with

$1 \leqq i_1 < \cdots < i_{d-m} \leqq n - m$. By Corollary 4.7, each of them has the coefficient $(d-m)!$. Hence the coefficient of $x_1 \cdots x_d$ in J_λ is

$$\frac{1}{(n-m)!}(d-m)!\binom{n-m}{d-m}d!(n-d)! = d!\,. \qquad \square$$

The next theorem establishes a direct relation between symmetric and non-symmetric Jack polynomials. It needs more variables than symmetrization but has the advantage of being stable in n. Observe, that λ is not required to be a partition.

4.10. Theorem. *Let $\lambda \in \Lambda$ and $m \in \mathbb{N}$ with $l(\lambda) \leqq m \leqq n - l(\lambda)$. Let λ^+ be the unique partition which is a permutation of $(\lambda_1, \ldots, \lambda_{n-m})$. Then*

$$J_{\lambda^+}(x_{m+1}, \ldots, x_n) = F_\lambda(0, \ldots, 0, x_{m+1}, \ldots, x_n)\,.$$

Proof. Recall that $\mathscr{P}_\lambda \subset \mathscr{P}_\alpha$ is the $\mathbb{Q}(\alpha)$-linear subspace spanned by all $E_{w\lambda}$, $w \in W$. Then Corollary 3.3 implies that $\mathscr{P}_\lambda|_{x_{n+1-m}=\cdots=x_n=0} = \mathscr{P}_{\lambda^+} \subseteq \mathbb{Q}(\alpha)[x_1, \ldots, x_{n-m}]$. Since $\mathscr{P}_\lambda$ is W-stable we conclude that also $\mathscr{P}_\lambda|_{x_1=\cdots=x_m=0} = \mathscr{P}_{\lambda^+} \subseteq \mathbb{Q}(\alpha)[x_{m+1}, \ldots, x_n]$. Lemma 2.3 implies that both sides of the equation are equal up to a factor $c \in \mathbb{Q}(\alpha)$. To determine c we may assume that $n \geqq m + |\lambda|$. Then, by Corollaries 4.7 and 4.9, the monomial $x_{m+1} \cdots x_{m+|\lambda|}$ figures on both sides with the same non-zero coefficient. Hence $c = 1$. $\square$

Although, as already indicated in the introduction, the Macdonald conjecture follows immediately from the combinatorial formula of the next section, a direct proof using the recursion formula might be of interest. To formulate its analogue for the non-symmetric polynomials we introduce the following notation. Fix an $m \in \mathbb{N}$ with $0 \leqq m \leqq n$. We split every $\lambda \in \Lambda$ in two parts λ' and λ'' where λ' (respectively λ'') consists of the first m (respectively last $n - m$) components of λ. We write $\lambda = \lambda'\lambda''$. Then we define the partially symmetric monomial functions as $m_\lambda^{(m)} := \sum_\mu x^{\lambda'\mu}$ where μ runs through all permutations of λ''. Their augmented version is $\tilde{m}_\lambda^{(m)} := u_{\lambda''} m_\lambda^{(m)}$. Let $\Lambda^{(m)} \subseteq \Lambda$ be the set of those λ where λ'' is a partition. Observe that $\Lambda^{(0)} = \Lambda^+$, $m_\lambda^{(0)} = m_\lambda$, and $\tilde{m}_\lambda^{(0)} = \tilde{m}_\lambda$.

4.11. Theorem. a) *Let $\lambda \in \Lambda$ and $m \in \mathbb{N}$ with $m \geqq l(\lambda)$. Then*

$$F_\lambda(x;\alpha) = \sum_{\mu \in \Lambda^{(m)}} a_{\lambda\mu}(\alpha)\tilde{m}_\mu^{(m)}$$

with $a_{\lambda\mu} \in \mathbb{N}[\alpha]$ for all $\mu \in \Lambda^{(m)}$.

b) *Let $\lambda \in \Lambda^+$. Then $J_\lambda(x;\alpha) = \sum_{\mu \in \Lambda^+} b_{\lambda\mu}(\alpha)\tilde{m}_\mu$ with $b_{\lambda\mu} \in \mathbb{N}[\alpha]$ for all $\mu \in \Lambda^+$.*

Proof. Part b) follows immediately from a) and Theorem 4.10. The proof of a) is by induction on $|\lambda|$. First observe that it suffices to prove the theorem for $m = l(\lambda)$. Since $|\lambda^*| = |\lambda| - 1$ and $l(\lambda^*) \leqq m$, the assertion is true for F_{λ^*}. With $\Psi := \Phi_{m+1} + \cdots + \Phi_n$ we have $Y_\lambda = (\bar{\lambda}_m + m)\Phi_m + \Psi$. Moreover $\bar{\lambda}_m + m = \alpha\lambda_i - k + m$ where k is the number of $j = 1, \ldots, m-1$ with $\lambda_j \geqq \lambda_m$. Thus $-k + m \geqq 1$. By the recursion formula Theorem 4.6, it suffices to prove the following.

Claim. Let $\mu \in \Lambda^{(m)}$. Then both $\Phi_m(\tilde{m}_\mu^{(m)})$ and $\Psi(\tilde{m}_\mu^{(m)})$ are linear combinations of $\tilde{m}_\nu^{(m)}$, $\nu \in \Lambda^{(m)}$ with coefficients in $\mathbb{N}$.

The effect of Φ_i on monomials is $\Phi_i(x^\nu) = x^{\bar{\nu}}$ where

$$\bar{\nu} := (\nu_2, \ldots, \nu_i, \nu_1 + 1, \nu_{i+1}, \ldots, \nu_n) .$$

In particular, Φ_m affects only the first m variables which proves the claim for Φ_m.

It is easy to check that the Φ_i satisfy the following commutation relations:

$$\begin{aligned} s_j\Phi_i &= \Phi_i s_j, \quad \text{if } i < j \\ s_j\Phi_j &= \Phi_{j+1} , \\ s_j\Phi_{j+1} &= \Phi_j , \\ s_j\Phi_i &= \Phi_i s_{j+1}, \quad \text{if } i > j+1 \end{aligned}$$

This shows that $\Psi(\tilde{m}_\mu^{(m)})$ is invariant for $s_{m+1}, \ldots, s_n$. In particular, it suffices to check the coefficient of $x^{\bar{\nu}}$ in $\Psi(\tilde{m}_\mu^{(m)})$ when $\bar{\nu} \in \Lambda^{(m)}$.

Assume that x^ν occurs in $m_\mu^{(m)}$, i.e., that $\nu' = \mu'$ and that ν'' is a permutation of μ''. Then ν is recovered from $\bar{\nu}$ by removing a part $\bar{\nu}_i$ of $\bar{\nu}$ with $\bar{\nu}_i = k := \mu_1 + 1$ and $i \geqq m$ and putting μ_1 in front. This shows that $\Psi(m_\mu^{(m)})$ contains $x^{\bar{\nu}}$ with multiplicity $m_k(\bar{\nu}'')$. Furthermore, $m_i(\bar{\nu}'') \leqq m_i(\mu'')$ for $i \neq k$ and $m_k(\bar{\nu}'') \leqq m_k(\mu'') + 1$. This implies that $\Psi(\tilde{m}_\mu^{(m)})$ contains $\tilde{m}_\nu^{(m)}$ with positive integral multiplicity. □

5. The combinatorial formula

In this section we give a simple and explicit formula for both the symmetric and non-symmetric Jack polynomials. Let $\Lambda \in \Lambda$. A *generalized tableau* of shape λ is a labelling T of the diagram of λ by the numbers $1, \ldots, n$. The *weight* of T is $|T| = (|T|_1, \ldots, |T|_n)$ where $|T|_i$ is the number of occurrences of the label i in T. Of course $|T|$ is S_n-conjugate to a unique partition. One writes x^T for the monomial $x^{|T|}$.

Definition. *A generalized tableau of shape $\lambda \in \Lambda$ is* **admissible** *if for all $(i,j) \in \lambda$*

a) $T(i,j) \neq T(i',j)$ *if* $i' > i$.

b) $T(i,j) \neq T(i',j-1)$ *if* $j > 1$, $i' < i$.

It is called 0-**admissible** *if additionally*
c) $T(i,j) \in \{i, i+1, \ldots, n\}$ *if* $j = 1$.

Definition. *Let T be a generalized tableau of shape λ.*
a) *A point $(i,j) \in \lambda$ is called* **critical** *if $j > 1$ and $T(i,j) = T(i,j-1)$.*
b) *The point $(i,j) \in \lambda$ is called* 0-**critical** *if it is critical or $j = 1$ and $T(i,j) = i$.*

The *hook-polynomials* of T are

$$d_T(\alpha) := \prod_{s \text{ critical}} d_\lambda(s,\alpha)\ ;$$

$$d_T^0(\alpha) := \prod_{s\ 0\text{-critical}} d_\lambda(s,\alpha)\ .$$

Our terminology can be explained as follows. Consider the tableau T^0 which arises from T by adding a zero-th column and labelling its boxes consecutively by $1, 2, \ldots, n$. Then T is 0-admissible if T^0 is admissible and a box s in T is 0-critical if it is critical in T^0.

Our main theorem is:

5.1. Theorem. *Let $\lambda \in \Lambda$. Then*

$$F_\lambda(x;\alpha) = \sum_{T\ 0\text{-admissible}} d_T^0(\alpha) x^T\ .$$

Let $\lambda^+ \in \Lambda^+$ be the unique partition conjugated to λ. Then

$$J_{\lambda^+}(x;\alpha) = \sum_{T \text{ admissible}} d_T(\alpha) x^T\ .$$

Proof. We prove first the formula for J_{λ^+} assuming it for F_λ. Let $m := l(\lambda)$ and assume $n \geqq |\lambda| + l(\lambda)$. Consider only those tableaux of shape λ which contain only labels $> m$. Then "0-admissible", "0-critical" are the same as "admissible", "critical" respectively. By Theorem 4.10, the formula for F_λ implies that for J_{λ^+}.

For the non-symmetric case, denote the right hand side of the formula by F'_λ. We are going to prove the following two lemmas.

5.2. Lemma. *Suppose $\lambda_i = 0$ and $\lambda_{i+1} > 0$, and write $d := d_\lambda(\alpha, (i+1, 1))$. Then we have $dF'_{s_i\lambda} = (d-1)s_i(F'_\lambda) + F'_\lambda$.*

For $\lambda \in \Lambda$ let $\Phi(\lambda) := (\lambda_2, \ldots, \lambda_n, \lambda_1 + 1)$.

5.3. Lemma. *Let $d := d_{\Phi\lambda}(\alpha, (n, 1))$ then $F'_{\Phi\lambda} = d\Phi(F'_\lambda)$.*

We finish first the proof of Theorem 5.1. In the situation of Lemma 5.2 let $\mu := s_i\lambda$. Then $d_\mu(i, 1) = d - 1$ while the hook-length of the remaining boxes doesn't change. Hence, if $F_\lambda = cE_\lambda$ then $F_\mu = \frac{d-1}{d} cE_\mu$. Let a_0 be the number of $k = i+2, \ldots, n$ with $\lambda_k > 0$. Then $\bar{\mu}_{i+1} = -i - a_0$ while $\bar{\mu}_i = d - 1 - i - a_0$ (Lemma 4.5). Hence, $x := \bar{\mu}_i - \bar{\mu}_{i+1} = d - 1$. With Proposition 4.3

we get $dF_\mu = (d-1)cE_\mu = xcE_\mu = (xs_i+1)cE_\lambda = ((d-1)s_i+1)F_\lambda$. We conclude from Lemma 5.2 that $F_\lambda = F'_\lambda$ implies $F_{s_i\lambda} = F'_{s_i\lambda}$.

In the same manner, we obtain from Lemma 5.3 and Corollary 4.2 that $F_{\Phi\lambda} = F'_{\Phi\lambda}$ if and only if $F_\lambda = F'_\lambda$. Since every $\lambda \in \Lambda$ is obtained by repeatedly applying Φ or switching a zero and a non-zero entry, the theorem follows by induction (and $F_0 = F'_0 = 1$). □

Proof of Lemma 5.2. If T is a tableau of shape λ, let T' be the tableau of shape $s_i\lambda$ obtained by moving all the points in row $i+1$ up one unit to the previously empty row i.

Let us ignore for a moment the labels of $(i+1,1) \in \lambda$ and $(i,1) \in s_i\lambda$. For all *other* points in T, the label is admissible (resp. critical) if and only it is so for the corresponding point in T', and the twisted hooklengths are unchanged. (In fact, l', l'' and a_λ are all unchanged!)

To examine the contributions of $(i,1)$ and $(i+1,1)$, we divide admissible tableaux T of shape λ into two classes:
$A = \{T \mid T(i+1,1) \neq i+1\}$, and $B = \{T \mid T(i+1,1) = i+1\}$.

Similarly we divide admissible tableaux U of shape $s_i\lambda$ into three classes:
$A' = \{U \mid U(i,1) \neq i, i+1\}$, $B' = \{U \mid U(i,1) = i+1\}$, and $B'' = \{U \mid U(i,1) = i\}$.

The map $T \mapsto T'$ is a bijection from A to A', and satisfies $d_T(\alpha)x^T = d_{T'}(\alpha)x^{T'}$. Also, if $T \in A$ then replacing each occurrence of the label i by $i+1$ and vice versa, we get another tableau $s_iT \in A$ with $d_T(\alpha) = d_{s_iT}(\alpha)$. This implies

$$\sum_{U \in A'} d_U(\alpha)x^U = \sum_{T \in A} d_T(\alpha)x^T = s_i \sum_{T \in A} d_T(\alpha)x^T \ .$$

$T \mapsto T'$ is also a bijection from B to B', however $T(i+1,1)$ is critical but $T'(i,1)$ is not. Since $d_\lambda(\alpha,(i+1,1)) = d$, we get

$$d \sum_{U \in B'} d_U(\alpha)x^U = \sum_{T \in B} d_T(\alpha)x^T \ .$$

Finally $T \mapsto s_iT'$ is a bijection from B to B'', and $T(i+1,1)$, $s_iT'(i,1)$ are both critical. Since $d_{s_i\lambda}(\alpha,(i,1)) = d-1$ (l'', a_λ are unchanged, while l' decreases by 1), we get

$$d \sum_{U \in B''} d_U(\alpha)x^U = (d-1)s_i \sum_{T \in B} d_T(\alpha)x^T \ .$$

Combining these we get $dF'_{s_i\lambda} = d\sum_{A'} + d\sum_{B'} + d\sum_{B''} = [\sum_A + (d-1)s_i\sum_A] + \sum_B + (d-1)s_i\sum_B = (d-1)s_iF'_\lambda + F'_\lambda$. □

Proof of Lemma 5.3. For a tableau T of shape λ, let T' be the tableau of shape $\Phi\lambda$ constructed as follows:

1) move rows 2 through n up one place.
2) prefix the first row by a point with the label 1 and move the row to the n-th place.
3) modify the labels by changing all 1's to n's and the other i's to $(i-1)$'s.

If s is a point in T, we write s' for the corresponding point in T', thus $s = (1, j)$ corresponds to $s' = (n, j+1)$ and for $i > 1$, $s = (i, j)$ corresponds to $s' = (i-1, j)$.

First observe the twisted hooklengths of corresponding points are the same. Indeed $a_\lambda(s) = a_{\Phi\lambda}(s')$, and $l'_\lambda(s) + l''_\lambda(s) = l'_{\Phi\lambda}(s') + l''_{\Phi\lambda}(s')$. ($l'$ might decrease by 1, but then l'' increases by 1, so that the sum is unchanged.)

Second, note that if T is admissible then so is T'. This is obvious for the first column, and for (i, j) with $j > 1$ and $i < n$, we only need to check that $T'(i, j) \neq T'(n, j)$. But these labels are obtained by applying 3) to the labels $T(i+1, j)$ and $T(1, j-1)$ which are distinct by the admissibility of T. The argument for the admissibility of $T'(n, j)$ is similar.

Next, note that the map $T \mapsto T'$ is actually a bijection from admissible tableaux of shape λ to those of shape $\Phi\lambda$. The inverse map is obtained by deleting the label $T'(n, 1)$ (which must be n), moving the last row to the top, and applying the inverse of 3).

Now, observe that the point $(n, 1)$ is a critical point of T', and any other point s' of T' is critical if and only if the corresponding point s in T is critical. This is obvious for all points except $(n, 2)$ which corresponds to $(1, 1)$ in T; but $T'(n, 2) = T'(n, 1) = n$ if and only if $T(1, 1) = 1$.

Finally by 2) and 3), if the weight of T is μ then the weight of T' is $\Phi\mu$, thus $x^{T'} = \Phi(x^T)$. This means

$$F'_{\Phi\lambda} = \sum_{T'} d_{T'}(\alpha) x^{T'} = d_{\Phi\lambda}(\alpha, (n, 1)) \sum_{T} d_T(\alpha) \Phi(x^T) = d \Phi F'_\lambda.$$

□

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