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Difference Equations and Symmetric Polynomials Defined by Their Zeros

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1 Introduction

In this paper, we are starting a systematic analysis of a class of symmetric polynomials which, in full generality, was introduced in [Sa]. The main features of these functions are that they are defined by vanishing conditions and that they are nonhomogeneous. They depend on several parameters, but we are studying mainly a certain subfamily which is indexed by one parameter, r. As a special case, we obtain for $r = 1$ the factorial Schur functions discovered by Biedenharn and Louck [BL].

Our main result is that for general r these functions are eigenvalues of difference operators, which are difference analogues of the Sekiguchi-Debiard differential operators. Thus the functions under investigation are nonhomogeneous variants of Jack polynomials.

More precisely, consider the set of partitions of length n, i.e., sequences of integers (λ_i) with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The weight $|\lambda|$ of a partition λ is the sum of its parts λ_i. Choose a vector $\rho \in \mathbb{C}^n$ which has to satisfy a mild condition. Then, for every λ, there is (up to a constant) a unique symmetric polynomial P_λ of degree at most d which satisfies the following vanishing condition:

$$P_\lambda(\mu + \rho) = 0 \text{ for all partitions } \mu \text{ with } |\mu| \leq |\lambda| \text{ and } \mu \neq \lambda.$$

This kind of vanishing comes up in the study of invariant differential operators and Capelli-type identities on multiplicity-free spaces and has been, in special cases, observed by other authors (e.g., [HU], [Ok]).

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In full generality, we have basically only one result (beyond their existence) about the polynomials P_λ, namely, two explicit formulas for P_λ when $\lambda = 1^k$. From then on, we only consider $\rho = r\delta$, where $r \in \mathbb{C}$ and $\delta = (n-1, n-2, \ldots, 1, 0)$.

We prove that these P_λ are simultaneous eigenfunctions of n commuting *difference* operators. On the highest homogeneous part of a polynomial, these difference operators act like well-known differential operators: the Sekiguchi-Debiard operators. The eigenfunctions of those are the Jack polynomials. This has as immediate consequence that the top homogeneous part of P_λ is a Jack polynomial.

In the later sections, we draw several conclusions from the difference equations. As an application to the "classical" theory, we give a new proof of the Pieri rule for Jack polynomials using the polynomials P_λ.

We conclude with a brief discussion of the "integral" form J_λ, which, in the homogeneous case, is a rescaling of the P_λ by a certain hooklength factor. It turns out that the corresponding *inhomogeneous* polynomial seems to have integrality and positivity properties which generalize a conjecture of Macdonald for the homogeneous case. In this connection, we have recently proved some integrality and positivity results which we shall report on elsewhere.

2 The basic construction

The results of this section are essentially in [Sa]. However, in order to keep the development self-contained, we give a quick rederivation.

Let us write $S(n, d) \subset \mathbb{Z}^n$ for the set of partitions $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ with $|\lambda| := \sum \lambda_i = d$. We say that $\rho \in \mathbb{C}^n$ is *dominant* if $\rho_i - \rho_j \neq -1, -2, -3, \ldots$ for all $i < j$. Slightly weakening this condition, we define ρ to be d-*dominant* if $\rho_i - \rho_j \neq -1, -2, -3, \ldots, -\lfloor d/i \rfloor$ for all $i < j$ where $d \in \mathbb{N}$.

Theorem 2.1. For any $d \in \mathbb{N}$ and $\rho \in \mathbb{C}^n$, put $M := S(n, d) + \rho \subseteq \mathbb{C}^n$. Assume ρ is d-dominant. Then, for every map $\bar{f}\colon M \to \mathbb{C}$, there is a unique symmetric polynomial f of degree at most d such that $f|_M = \bar{f}$. □

Proof. For any partition $\lambda \in \mathbb{Z}^n$, let m_λ be the corresponding monomial symmetric function in n variables. If we express an arbitrary symmetric function of degree $\leq d$ in terms of m_λ, then the interpolation problem gives a *square* system of linear equations for the coefficients. Hence existence implies uniqueness.

To show existence, we argue by induction on $n + d$. The case $n = 0$ is vacuous, so we assume $n \geq 1$.

To any $\lambda \in S(n-1, d)$ we can append a zero and obtain a partition $\lambda, 0 \in S(n, d)$.

This way, we can define a map $g = \sum a_\lambda m_\lambda \mapsto g^+ = \sum a_\lambda m_{\lambda,0}$. It is an injective map from symmetric functions in $n-1$ variables to symmetric functions in n variables. It has the property that g^+ has the same degree as g, and $g^+(x_1, \dots, x_{n-1}, 0) = g(x_1, \dots, x_{n-1})$.

We will construct f as a function of the form

$$f(x) = g^+(x_1 - \rho_n, \dots, x_n - \rho_n) + \left[\prod_{i=1}^{n} (x_i - \rho_n)\right] h(x_1 - 1, \dots, x_n - 1)\,.$$

First, let us consider the set M_0 of all points $x = \lambda + \rho \in M$ with $\lambda_n = 0$. Since $x_n - \rho_n = 0$, the first term equals $g(x_1 - \rho_n, \dots, x_{n-1} - \rho_n)$ and the second term vanishes. If x runs through M_0, then $x' = (x_1 - \rho_n, \dots, x_{n-1} - \rho_n)$ runs through $S(n-1, d) + \rho'$, where $\rho' := (\rho_1 - \rho_n, \dots, \rho_{n-1} - \rho_n)$, which is also d-dominant. By induction, we can find g of degree $\le d$ with $f(x) = g(x') = \overline{f}(x)$ for all $x \in M_0$.

Next, we consider the points $x \in M \setminus M_0$, i.e., $x = \lambda + \rho \in M$ with $\lambda_n > 0$. These exist only if $d \ge n$. As x runs through these points, $(x_1 - 1, \dots, x_n - 1)$ will run through $S(n, d-n) + \rho$. Since $\lfloor d/i \rfloor \ge \lambda_i \ge \lambda_n > 0$ and since ρ is d-dominant, each of the factors $x_i - \rho_n = \lambda_i + \rho_i - \rho_n$ is nonzero. By induction, we can find h of degree $\le d - n$ such that h has prescribed values at $M \setminus M_0$. ■

We assume from now on that ρ is dominant. With the theorem, we are going to define interpolation polynomials. To get the most convenient normalization, we have to introduce some more notation: Recall that a partition λ can be represented by its *diagram*, i.e., the set of all lattice points (called boxes) $(i, j) \in \mathbb{Z}^2$ with $1 \le i \le n$ and $1 \le j \le \lambda_i$. The dual partition λ' is the one with the transposed diagram. Now, for every box s, we define the ρ-*hooklength* to be $c_\lambda^\rho(s) := (\lambda_i - j + 1) + (\rho_i - \rho_{\lambda'_j})$ and $c_\lambda^\rho := \prod_{s\in\lambda} c_\lambda^\rho(s)$.

Definition. For any partition $\lambda \in S(n, d)$, let P_λ^ρ be the unique polynomial in n variables such that

(1) P_λ^ρ is symmetric;

(2) $\deg P_\lambda^\rho \le d$;

(3) $P_\lambda^\rho(\mu + \rho) = 0$ for all $\mu \in S(n, d)$, $\mu \ne \lambda$;

(4) $P_\lambda^\rho(\lambda + \rho) = c_\lambda^\rho$.

The normalization condition (4) is motivated by the following theorem. In fact, we could replace (4) by it.

Theorem 2.2. Let $P_\lambda^\rho = \sum_{\mu:\, |\mu| \le |\lambda|} u_{\lambda\mu}^\rho m_\mu$ be the expression in terms of monomial symmetric functions. Then $u_{\lambda\lambda}^\rho = 1$. □

Proof. We proceed by induction on $n + |\lambda|$. As in the proof of Theorem 2.1, we express

$$P_\lambda^\rho = g^+(x_1 - \rho_n, \dots, x_n - \rho_n) + \left[\prod_{i=1}^{n} (x_i - \rho_n)\right] h(x_1 - 1, \dots, x_n - 1)\,.$$

First assume $\lambda_n = 0$. Put $\nu := (\lambda_1, \ldots, \lambda_{n-1})$ and $\rho' := (\rho_1 - \rho_n, \ldots, \rho_{n-1} - \rho_n)$. Then Theorem 2.1 implies $g = aP^{\rho'}_\nu$ with $a \in \mathbb{C}^*$. Now we compare values at $x = \lambda + \rho$. Since $c^\rho_\lambda = c^{\rho'}_\nu$, we obtain $a = 1$ and the assertion follows by induction.

Next, suppose $\lambda_n > 0$. Then Theorem 2.1 implies $g = 0$ and $h = aP^\rho_\nu(x_1 - 1, \ldots, x_n - 1)$ where $\nu := (\lambda_1 - 1, \ldots, \lambda_n - 1)$ and $a \in \mathbb{C}^*$. Again, we compare values at $x = \lambda + \rho$. The linear factors are just the ρ-hooklengths for the first column of λ. Thus, $a = 1$ and the assertion follows by induction. ■

Additionally, we get the following reduction formula.

Corollary 2.3. Assume λ is a partition with $\lambda_n > 0$, and let $\lambda^* := (\lambda_1 - 1, \ldots, \lambda_n - 1)$. Then $P^\rho_\lambda = \prod_i (x_i - \rho_n) P^\rho_{\lambda^*}(x_1 - 1, \ldots, x_n - 1)$. □

3 Special cases

We do not know an explicit formula for P^ρ_λ in general, but several special cases are known.

For arbitrary ρ we have only a formula for $\lambda = 1^k$. This is the partition with k ones and $(n - k)$ zeros. The functions $P^\rho_{1^k}$ are important since they are analogues of the elementary symmetric functions. In particular, they generate the symmetric polynomials as a ring. Actually, we have *two* formulas for them.

Recall that the elementary symmetric function $e_j(x)$ and the complete symmetric function $h_j(y)$ are the coefficients of t^j in the expansions of $E(x, t) = \prod_i (1 + tx_i)$ and $H(y, t) = \prod_i (1 - ty_i)^{-1}$, respectively.

Proposition 3.1. Let ρ be dominant and $1 \le k \le n$. Then

$$P^\rho_{1^k} = \sum_{j=0}^{k} (-1)^{k-j} h_{k-j}(\rho_k, \ldots, \rho_n) e_j(x) = \sum_{i_1 < \cdots < i_k} \prod_{j=1}^{k} (x_{i_j} - \rho_{i_j + k - j}). \qquad \square$$

Proof. Denote the first expression by P', and the second by P''. We are going to show that they both satisfy the definition of $P^\rho_{1^k}$. Both have certainly the right degree and m_{1^k} has the right coefficient.

For the vanishing condition (3), let $x = \mu + \rho$ with $|\mu| \le k$ and $\mu \ne 1^k$. This forces $\mu_k = \cdots = \mu_n = 0$ and $x_k = \rho_k, \ldots, x_n = \rho_n$. Observe that P' is precisely the coefficient of t^k in the power series expansion of $\prod_{i=1}^n (1 + tx_i) / \prod_{i=k}^n (1 + t\rho_i)$. Evaluated at x, this quotient becomes a *polynomial* of degree $< k$, and its kth coefficient $P'(x)$ vanishes. As for P'', the index i_k in its definition is at least k. Hence the factors for $j = k$ vanish at x, which shows $P''(x) = 0$.

Finally, we have to show symmetry. This is trivial for P' but not quite for P''. First let $n = 2$. Then

$$P''_{1^1} = (x_1 - \rho_1) + (x_2 - \rho_2); \quad P''_{1^2} = (x_1 - \rho_2)(x_2 - \rho_2)\,,$$

which are certainly symmetric. Now let $n \geq 3$. To make the dependence on ρ and k visible, we write $P'' = P''_k(x;\rho)$. Furthermore, let x', ρ' (resp. x'', ρ'') equal x, ρ where we dropped the last (resp. first) component. If we break the defining sum for P'' up according to whether $i_k < n$ or $i_k = n$, we get

$$P''_k(x;\rho) = P''_k(x';\rho') + (x_n - \rho_n)P''_{k-1}(x';\rho'').$$

By induction we see that P'' is symmetric in $x_1, \ldots, x_{n-1}$. If we break the sum up according to whether $i_1 = 1$ or not, we obtain

$$P''_k(x;\rho) = P''_k(x'';\rho'') + (x_1 - \rho_k)P''_{k-1}(x'';\rho'').$$

This shows that P'' is symmetric in $x_2, \ldots, x_n$ as well. ■

Remarks. For $\rho = r(n-1, \ldots, 1, 0)$, the expression P' is essentially due to Wallach while that for P'' can be traced back to Capelli. The equality $P' = P''$ can be also proved directly by using the polynomials $e_k(x/y)$ of [M3, p. 58].

For the rest of the paper we specialize to ρ of the form $r\delta$, where r is a complex number or just an indeterminate and $\delta := (n-1, \ldots, 1, 0)$. The dominance of ρ means that $r \neq -p/q$ where p, q are integers such that $p, q \geq 1$, and $q < n$. We shall assume this from now on.

First we treat the case $r = 0$. For this we introduce the *falling factorial polynomials* $x^{\underline{m}} := x(x-1)\cdots(x-m+1)$. The factorial monomial symmetric functions $m_{\underline{\lambda}}$ are obtained by replacing each monomial $x_1^{l_1}x_2^{l_2}\ldots x_n^{l_n}$ in m_λ by the corresponding factorial monomial $x_1^{\underline{l_1}}x_2^{\underline{l_2}}\ldots x_n^{\underline{l_n}}$. The following is obvious.

Proposition 3.2. For $r = 0$, we have $P^0_\lambda = m_{\underline{\lambda}}$. □

For $r = 1$ we get the factorial Schur functions. (See [BL], [M2], and [Ol].) To define them, we write $a_\delta(x)$ for the Vandermonde determinant $\det(x_i^{\delta_j}) = \prod_{i<j}(x_i - x_j)$. Then the next result seems to be due to Okounkov [Ok].

Proposition 3.3. For $r = 1$, we have

$$P^\delta_\lambda(x) = \frac{1}{a_\delta(x)} \det\left(x_i^{\underline{\lambda_j + \delta_j}}\right). \quad \square$$

Proof. Since $\det(x_i^{\underline{\lambda_j+\delta_j}})$ is a skew-symmetric polynomial, its quotient by a_δ is a symmetric polynomial which is easily seen to have degree $|\lambda|$. Now let $\mu \neq \lambda$ and $|\mu| \leq |\lambda|$. Since $a_\delta(\mu+\delta) \neq 0$ for any partition μ, it remains only to prove the vanishing of $\det[(\mu_i+\delta_i)^{\underline{\lambda_j+\delta_j}}] = \sum_\sigma (-1)^\sigma \prod_i (\mu_{\sigma(i)} + \delta_{\sigma(i)})^{\underline{\lambda_i+\delta_i}}$.

If a, b are nonnegative integers, then $a^{\underline{b}} = 0$ unless $a \geq b$. So the σ-summand vanishes unless $\mu_{\sigma(i)} + \delta_{\sigma(i)} \geq \lambda_i + \delta_i$ for all i. Summing over i, we observe that $|\mu| \leq |\lambda|$ forces equality for each i, which implies $\sigma(\mu+\delta) = \lambda+\delta$. But this is not possible for $\mu \neq \lambda$. ■

Finally we consider the analogue of the complete symmetric functions, i.e., $P_d^{r\delta}$ where d stands for $(d, 0, \ldots, 0)$.

Proposition 3.4. For $d \geq 0$ we have

$$P_d^{r\delta} = \binom{-r}{d}^{-1} \sum_{i_j} \prod_{j=1}^{n} \left[\binom{-r}{i_{j-1} - i_j} (x_j - r\delta_j - i_j)^{\underline{i_{j-1} - i_j}} \right]$$

where the sum runs through all integer sequences $d = i_0 \geq i_1 \geq \cdots \geq i_{n-1} \geq i_n = 0$. □

Proof. Let p_d denote the right-hand side. Obviously, it has the right degree d, and the coefficient of x_1^d is one. Next we show that the vanishing condition holds. For this, let $x = \mu + r\delta$ with $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. Then every summand of p_d is a multiple of $y_1(y_2 - 1)\cdots(y_d - d + 1)$ where $y_1 = \cdots = y_{i_{n-1}} = x_n - r\delta_n = \mu_n$, $y_{i_{n-1}+1} = \cdots = y_{i_{n-2}} = \mu_{n-1}$, etc. In particular, the y_i are integers with $0 \leq y_1 \leq \cdots \leq y_d \leq \mu_1$. Now assume that the product does not vanish, i.e., $y_i \neq i - 1$ for all i. Then we claim $y_i \geq i$ for all i. Indeed, $y_i \geq y_{i-1} \geq i - 1$ and $y_i \neq i - 1$ imply $y_i \geq i$. In particular, $\mu_1 \geq y_d \geq d$. But this is not possible for our choice of μ. This shows $p_d(x) = 0$.

Finally, we have to prove symmetry. We are considering the case $n = 2$ first. For this we need two basic facts about falling factorials:

(1) $x^{\underline{a}}(x - a)^{\underline{b}} = x^{\underline{a+b}}$ (which is obvious) and

(2) $(x + y)^{\underline{n}} = \sum_{i=0}^{n} \binom{n}{i} x^{\underline{i}} y^{\underline{n-i}}$ (the Vandermonde identity).

Letting $i_0 = d \geq i_1 = i \geq i_2 = 0$, we obtain that p_d is a multiple of

$$\sum_i \binom{-r}{d-i} (x_1 - r - i)^{\underline{d-i}} \binom{-r}{i} x_2^{\underline{i}}.$$

Applying identity (2), this becomes

$$\sum_{i,j} \frac{(d-i)!(-r)^{\underline{d-i}}(-r)^{\underline{i}}(-r-i)^{\underline{d-i-j}}}{j!(d-i-j)!(d-i)!i!} x_1^{\underline{j}} x_2^{\underline{i}}.$$

Using (1), the coefficient becomes $(-r)^{\underline{d-i}}(-r)^{\underline{d-j}} / j!(d-i-j)!i!$, which implies symmetry for $p_d(x_1, x_2)$.

Now suppose that $n \geq 3$. Summing over $i = i_{n-1}$ first, we obtain

$$p_d(x) = \binom{-r}{d}^{-1} \sum_{i=0}^{d} \binom{-r}{d-i}\binom{-r}{i} x_n^{\underline{i}} p_{d-i}(x_1 - r - i, \dots, x_{n-1} - r - i)\,.$$

By induction we conclude that p_d is symmetric in $\{x_1, \dots, x_{n-1}\}$. Summing over $i = i_1$, we obtain

$$p_d(x) = \binom{-r}{d}^{-1} \sum_{i=0}^{d} \binom{-r}{d-i}\binom{-r}{i} (x_1 - r\delta_1 - i)^{\underline{d-i}} p_i(x_2, \dots, x_n)\,,$$

which proves symmetry in $\{x_2, \dots, x_n\}$. This concludes the proof. ■

4 Difference operators and Jack polynomials

In this section we deduce a different characterization of the polynomials $P_\lambda^{r\delta}$ in terms of difference equations.

Let ε_i be the ith canonical basis vector in $\mathbb{C}^n$. The ith *shift operator* T_i on functions is defined by $T_i f(x) := f(x - \varepsilon_i)$, and the ith *difference operator* is $\nabla_i := 1 - T_i$. These operators commute with each other, and T_i, ∇_i also commute with multiplication by x_j for $j \neq i$.

Definition. Let t be an indeterminate. For $1 \leq i, j \leq n$ put

$$\Delta_{ij} := (x_i + t)(x_i + r)^{\delta_j} - x_i^{\delta_j + 1} T_i, \quad \Delta := \det(\Delta_{ij}), \quad \mathcal{D}(t; r) := a_\delta(x)^{-1}\Delta.$$

Since Δ_{ij} and Δ_{kl} commute for $i \neq k$, the determinant Δ is well defined. Furthermore, it maps symmetric polynomials to skew-symmetric ones. Hence $\mathcal{D}(t; r)$ is a well-defined operator acting on the space of symmetric polynomials. We can develop

$$\mathcal{D}(t; r) = D_0 t^n + D_1 t^{n-1} + \cdots + D_n$$

into a polynomial where D_i is a difference operator of order i and $D_0 = 1$.

Example 4.1. For $r = 0$ we obtain $\mathcal{D}(t; r) = (t + x_1\nabla_1)\cdots(t + x_n\nabla_n)$, and hence $D_i = e_i(x_1\nabla_1, \dots, x_n\nabla_n)$.

We need the following partial order relation on $\mathbb{Z}^n$: we say $\mu \leq \lambda$ if $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all $1 \leq i \leq n$. It has the property that λ is a partition if and only if it is maximal among all its permutations.

Lemma 4.2. The operator $\mathcal{D}(t; r)$ is triangular. More precisely,

$$\mathcal{D}(t; r) m_\lambda \in \prod_i (\lambda_i + r\delta_i + t) m_\lambda + \sum_{\mu < \lambda} \mathbb{C}[t] m_\mu.$$

In particular, $\deg \mathcal{D}(t; r) f \leq \deg f$ for every symmetric polynomial f. □

Proof. The transition matrix between Schur function s_λ and monomial symmetric functions m_μ is unitriangular. Hence, it suffices to prove $\mathcal{D}(t;r)m_\lambda \in \prod_i(\lambda_i + r\delta_i + t)s_\lambda + \sum_{\mu<\lambda}\mathbb{C}[t]s_\mu$. Now we multiply by a_δ. By definition, $a_{\lambda+\delta} = a_\delta s_\lambda$ is the skew-symmetrization of $x^{\lambda+\delta}$. Therefore, it suffices to prove that Δm_λ is a linear combination of monomials x^μ with $\mu \le \lambda + \delta$ and that the coefficient of $x^{\lambda+\delta}$ has the indicated form.

For this, observe $\Delta_{ij} = x_i^{\delta_j}(x_i\nabla_i + r\delta_j + t) +$ lower terms in x_i, and that $x_i\nabla_i(x_i^m) = mx_i^m +$ lower terms. Thus

$$\Delta_{ij}x_i^m = (m + r\delta_j + t)x_i^{m+\delta_j} + \text{ lower terms in } x_i.$$

Expanding the determinant defining Δ, we see that all monomials occurring in Δm_λ are of the form x^μ with $\mu = \sigma(\lambda) + \tau(\delta) - \eta$, where σ, τ are permutations and $\eta \in \mathbb{N}^n$. All these μ are $\le \lambda + \delta$. Furthermore, $\mu = \lambda + \delta$ implies $\sigma(\lambda) = \lambda$, $\tau = 1$, and $\eta = 0$. In particular, only the diagonal term contributes to $x^{\lambda+\delta}$. Hence, we obtain

$$\Delta m_\lambda \in \prod_i(\lambda_i + r\delta_i + t)x^{\lambda+\rho} + \sum_{\mu<\lambda+\rho}\mathbb{C}[t]x^\mu. \qquad \blacksquare$$

For $I \subseteq \{1, \ldots, n\}$, put $\varepsilon_I := \sum_{i\in I}\varepsilon_i$, and $T_If := (\prod_{i\in I}T_i)f = f(x - \varepsilon_I)$. Furthermore, we introduce the functions $\varphi_I(x) := \det c^I_{ij}(x)$ where

$$c^I_{ij} := \begin{cases} x_i^{\delta_j+1} & \text{for } i \in I; \\ (x_i + r)^{\delta_j} & \text{for } i \notin I. \end{cases}$$

They behave like "cutoff functions."

Lemma 4.3. Let $r \neq 0$ and μ be a partition. If $\mu - \varepsilon_I$ is not a partition, then $\varphi_I(\mu + r\delta) = 0$. □

Proof. Put $x = \mu + r\delta$ and assume $\mu - \varepsilon_I$ is not a partition. Then there are two cases:

(1) $\mu_n = 0$ and $n \in I$. Then $x_n = 0$ and the n-th row of $c^I(x)$ vanishes. Hence $\varphi_I(x) = 0$.

(2) There is $i < n$ such that $i \in I$, $i + 1 \notin I$, and $\mu_i = \mu_{i+1}$. In this case $x_i = x_{i+1} + r$ and c^I has two proportional rows. Hence, again $\varphi_I(x) = 0$ and the claim is proved. ■

Now we prove that each $P^{r\delta}_\lambda$ is an eigenfunction of $\mathcal{D}(t;r)$.

Theorem 4.4. For each partition λ, we have

$$\mathcal{D}(t;r)P^{r\delta}_\lambda = \prod_i(\lambda_i + r\delta_i + t)P^{r\delta}_\lambda.$$

In particular, the action of $\mathcal{D}(t;r)$ on symmetric polynomials is diagonalizable with distinct eigenvalues. □

Proof. In view of Lemma 4.2, it suffices to show that $\mathcal{D}(t;r)P_\lambda^{r\delta}$ satisfies the vanishing condition. We may exclude the case $r = 0$ either by direct computation or by continuity. Since, then, $a_\delta(\mu + r\delta) \neq 0$ for all partitions μ, we are left with $\Delta(f)$.

We can expand Δ as follows: $\Delta = \sum_I d_I T_I$, where $d_I = \det d^I_{ij}$ and

$$d^I_{ij} := \begin{cases} -x_i^{\delta_j+1} & \text{for } i \in I; \\ (x_i + t)(x_i + r)^{\delta_j} & \text{for } i \notin I. \end{cases}$$

Since d_I is a multiple of φ_I, Lemma 4.3 holds also for it. Let μ be a partition with $|\mu| \leq |\lambda|$, $\mu \neq \lambda$. Then $\Delta P_\lambda^{r\delta}(\mu + r\delta) = \sum_I d_I(\mu + r\delta)P_\lambda^{r\delta}(\mu - \varepsilon_I + r\delta)$. Since $P_\lambda^{r\delta}$ satisfies the vanishing condition, it follows from Lemma 4.3 that $d_I(\mu+r\delta)P_\lambda^{r\delta}(\mu-\varepsilon_I+r\delta) = 0$ for all I. This finishes the proof of the vanishing condition for $\mathcal{D}(t;r)P_\lambda^{r\delta}$ and of the theorem. ■

Since the $P_\lambda^{r\delta}$ form also an eigenbasis for $D_1, \ldots, D_n$ we obtain the following.

Corollary 4.5. The difference operators $D_1, \ldots, D_n$ commute pairwise. □

Corollary 4.6. Every $P_\lambda^{r\delta}$ has an expansion of the form $m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu$. □

Proof. Lemma 4.2 implies that $\mathcal{D}(t;r)$ preserves the finite-dimensional space spanned by $\{m_\mu \mid \mu \leq \lambda\}$. Thus, by the theorem, it has an eigenvector with the above expansion, which by the lemma has the same eigenvalue as $P_\lambda^{r\delta}$. So, they are equal. ■

Now we can make the connection to the Jack polynomials. First, we recall their definition: for an indeterminate t, consider the differential operators

$$\overline{\Delta} := \det\left(x_i^{\delta_j}(t + r\delta_j + x_i \frac{\partial}{\partial x_i}\right); \qquad \overline{\mathcal{D}}(t;r) := a_\delta^{-1}\overline{\Delta}.$$

These operators were introduced by Sekiguchi [Se] and Debiard [De]. Macdonald [M1] uses them to define the Jack polynomial $P_\lambda^{(1/r)}$: it is the unique eigenvector of $\overline{\mathcal{D}}(t;r)$ which is of the form $m_\lambda + \sum_{\mu<\lambda} a_\mu m_\lambda$.

Corollary 4.7. The top homogeneous component of $P_\lambda^{r\delta}$ is $P_\lambda^{(1/r)}$. □

Proof. Denote this component by $\overline{P}$. As observed in the proof of Lemma 4.2, $\Delta_{ij} = x_i^{\delta_j}(x_i\nabla_i + r\delta_j + X) +$ lower terms, and $x_i\nabla_i = x_i(\partial/\partial x_i) +$ lower terms. Thus $\mathcal{D}(t;r)$ acts on $\overline{P}$ by $a_\delta^{-1}\det(x_i^{\delta_j}(x_i(\partial/\partial x_i) + r\delta_j + t)) = \overline{\mathcal{D}}(t;r)$. Consequently, $\overline{P}$ is an eigenfunction of the Sekiguchi-Debiard operator. The assertion follows from Corollary 4.6. ■

5 The extra vanishing theorem

Corollary 4.6 states that $P_\lambda^{r\delta}$ contains fewer monomials than it could according to its definition. In this section we establish a property of $P_\lambda^{r\delta}$ which is in a way "dual" to that: we are going to prove that $P_\lambda^{r\delta}$ vanishes at more points than it should by definition.

Recall that $\lambda \subset \mu$ means $\lambda_i \leq \mu_i$ for all i, i.e., the diagrams are contained in each other. Let $\mathcal{P}$ be the set of partitions. A subset S of $\mathcal{P}$ is called *closed* if $\lambda \in S$, $\mu \in \mathcal{P}$, and $\lambda \subset \mu$ implies $\mu \in S$. For every closed set S, we consider the ideal $\mathcal{I}_S$ of symmetric polynomials which vanish at all points $\mu + r\delta$ where μ is a partition which is *not* in S.

Theorem 5.1. Let $S \subseteq \mathcal{P}$ be closed. Then the ideal $\mathcal{I}_S$ is stable under the action of $\mathcal{D}(t; r)$. □

Proof. Again, we may exclude $r = 0$ by continuity. Then we have to show that $\Delta(f)(x) = 0$ whenever $f \in \mathcal{I}_S$ and $x = \mu + r\delta$ with $\mu \in \mathcal{P} \setminus S$. As in the proof of Theorem 4.4 it suffices to consider the products $\varphi_I(x)f(x - \varepsilon_I)$. Assume this does not vanish. Then $\mu' = \mu - \varepsilon_I \in \mathcal{P}$ with $f(\mu' + r\delta) \neq 0$. But then $\mu' \in S$, and therefore $\mu \in S$, contradicting the choice of μ. ■

Now we can prove the extra vanishing theorem.

Theorem 5.2. Let λ and μ be partitions with $\lambda \not\subset \mu$. Then $P_\lambda^{r\delta}(\mu + \rho) = 0$. □

Proof. Consider the closed subset S of all μ containing λ. We have to show $P_\lambda^{r\delta} \in \mathcal{I}_S$. Now for generic r, there exist functions in $\mathcal{I}_S$ which are *nonzero* at $\lambda + r\delta$. (For example, the product of falling factorials $\prod_{i,j,k}(x_i - r\delta_j)^{\underline{\lambda_k}}$ is such a function.) The ideal $\mathcal{I}_S$ is $\mathcal{D}(t; r)$-stable. Since $\mathcal{D}(t; r)$ is diagonalizable, there must be an eigenfunction of $\mathcal{D}(t; r)$ in $\mathcal{I}_S$ with this property. But this function must be a multiple of some $P_\mu^{r\delta}$. Then $P_\mu^{r\delta}(\lambda + r\delta) \neq 0$ implies $|\mu| \leq |\lambda|$. Since $P_\mu^{r\delta}(\mu + r\delta) \neq 0$, we have $\lambda \subset \mu$. Hence $\mu = \lambda$. ■

This can be extended.

Corollary 5.3. Let $S \subseteq \mathcal{P}$ be closed. Then $\mathcal{I}_S = \bigoplus_{\lambda \in S} \mathbb{C}P_\lambda^{r\delta}$. □

Proof. Since $\mathcal{I}_S$ is $\mathcal{D}$-stable, there must be a $S' \subseteq \mathcal{P}$ with $\mathcal{I}_S = \oplus_{\lambda \in S'} \mathbb{C}P_\lambda^{r\delta}$. Let $\lambda \in S'$. Since $P_\lambda^{r\delta}(\lambda + r\delta) \neq 0$, it cannot be in $\mathcal{P} \setminus S$. Hence $S' \subseteq S$. Conversely, let $\lambda \in S$ and assume there is a $\mu \in \mathcal{P} \setminus S$ with $P_\lambda^{r\delta}(\mu + r\delta) \neq 0$. Then $\lambda \subset \mu$ by the extra vanishing theorem. Hence $\mu \in S$, which is impossible. This shows $S \subseteq S'$. ■

To round off this discussion, let us mention the following.

Proposition 5.4. Let Λ be the ring of symmetric polynomials (in n variables). Then every $\mathcal{D}$-stable ideal of Λ is of the form $\mathcal{I}_S$ for some closed subset S of $\mathcal{P}$. □

Proof. Clearly, every $\mathcal{D}$-stable ideal is of the form $\oplus_{\lambda \in S} \mathbb{C}P_\lambda^{r\delta}$. We have to show that S is closed. For this we need the following weak form of Pieri's rule proved in the next section: Let $e_1 = \sum_i x_i$. Expand $e_1 P_\lambda^{r\delta} = \sum_\mu a_\mu P_\mu^{r\delta}$. Then $a_\mu \neq 0$ whenever $\mu = \lambda + \varepsilon_i \in \mathcal{P}$. This implies $\mu = \lambda + \varepsilon_i \in S$ whenever $\lambda \in S$ and $\mu \in \mathcal{P}$, which is equivalent to S being closed. ■

6 The dehomogenization operators and the Pieri formula

Both the $P_\lambda^{r\delta}$ and the Jack polynomials $P_\lambda^{(1/r)}$ form a basis of the algebra Λ of symmetric polynomials. In particular, there is a linear isomorphism $\Psi\colon \Lambda \to \Lambda$ which maps $P_\lambda^{(1/r)}$ to $P_\lambda^{r\delta}$. We are going to show that Ψ can also be described in terms of difference operators.

For this we define the following variant of $\mathcal{D}$:

$$\mathcal{E} := a_\delta^{-1} \det[(x_i + r)^{\delta_j} + t x_i^{\delta_j+1} T_i] = 1 + \mathcal{E}_1 t + \cdots + \mathcal{E}_n t^n.$$

Let $\Lambda_d \subseteq \Lambda$ be the subspace spanned by all $P_\lambda^{r\delta}$ with $|\lambda| = d$. This is also the space of all polynomials of degree $\le d$ which vanish in all $\mu + r\delta$ with $|\mu| \le d - 1$.

Lemma 6.1. We have $\mathcal{E}_k(\Lambda_d) \subseteq \Lambda_{d+k}$. Moreover, the effect of $\mathcal{E}_k$ on the top homogeneous components is multiplication by the elementary symmetric function e_k. □

Proof. In the notation of Section 4, $\mathcal{E}_k$ has the expansion $\mathcal{E}_k = a_\delta^{-1} \sum_{|I|=k} \varphi_I T_I$. Hence $\mathcal{E}_k f(x) = a_\delta^{-1}(x) \sum_{|I|=k} \varphi_I(x) f(x - \varepsilon_I)$. Let $f \in \Lambda_d$ and μ be a partition with $|\mu| \le d + k - 1$ and $x = \mu + r\delta$. Then we have $\varphi_I(x) f(x - \varepsilon_I) = 0$. This means $\mathcal{E}_k f \in \Lambda_{d+k}$.

For the top homogeneous terms, $T_I = 1$ and $\varphi_I = \prod_{i\in I} x_i$, and hence $\mathcal{E}_k$ acts like multiplication by e_k. ■

Now we can prove the following.

Theorem 6.2. (a) The difference operators $\mathcal{E}_1, \ldots, \mathcal{E}_n$ commute pairwise.

(b) Let $\psi\colon \Lambda \to \mathbb{C}[\mathcal{E}_1, \ldots, \mathcal{E}_n]$ be the isomorphism with $\psi(e_k) = \mathcal{E}_k$. Then $\Psi(f) = \psi(f)(1)$ (evaluation at 1) for all $f \in \Lambda$. □

Proof. Let $\Lambda_{(d)}$ be the space of symmetric homogeneous polynomials of degree d. Then $\Psi\colon \Lambda_{(d)} \xrightarrow{\sim} \Lambda_d$, and the inverse is given by taking the top homogeneous component. Thus Lemma 6.1 implies that the following diagram commutes:

$$\begin{array}{ccc} \Lambda_{(d)} & \xrightarrow{\Psi} & \Lambda_d \\ \downarrow e_k & & \downarrow \mathcal{E}_k \\ \Lambda_{(d+k)} & \xrightarrow{\Psi} & \Lambda_{d+k}. \end{array}$$

Hence $\Psi(e_k f) = \mathcal{E}_k \Psi(f)$ for all $f \in \Lambda$. This shows (a). Let $f(x) = p(e_1, \ldots, e_k)$. Then $\Psi(f) = \Psi(p(e_k)) = p(\mathcal{E}_k)\Psi(1) = \psi(f)(1)$. ■

As an application of the theory above, we give a new proof of the Pieri rule for Jack polynomials.

At each lattice point $s = (i, j)$ in the diagram of λ, the *lower* and *upper* hooklengths are defined by $c_\lambda(s) = c_\lambda(\alpha; s) := \alpha(\lambda_i - j) + (\lambda'_j - i + 1)$, and $c'_\lambda(s) = c'_\lambda(\alpha; s) := \alpha(\lambda_i - j + 1) + (\lambda'_j - i)$.

Let $\mu \subset \lambda$. Then $X(\lambda/\mu)$ denotes the set of all boxes $(i, j) \in \lambda$ such that $\mu_i = \lambda_i$ and $\mu'_j < \lambda'_j$. Then we define

$$\psi'_{\lambda/\mu}(\alpha) := \prod_{s \in X(\lambda/\mu)} \frac{c_\lambda(\alpha; s)/c'_\lambda(\alpha; s)}{c_\mu(\alpha; s)/c'_\mu(\alpha; s)}.$$

The Pieri formula is the following identity.

Theorem 6.3. For every partition μ, we have $e_k P_\mu^{(\alpha)} = \sum_\lambda \psi'_{\lambda/\mu}(\alpha) P_\lambda^{(\alpha)}$ where λ runs over all partitions of the form $\mu + \varepsilon_I$ for some $I \subset \{1, \ldots, n\}$ with $|I| = k$, i.e., $\lambda - \mu$ is a vertical k-strip. □

Proof. Applying Ψ to both sides, it suffices to prove $\mathcal{E}_k P_\mu^{r\delta} = \sum_\lambda \psi'_{\lambda/\mu}(1/r) P_\lambda^{r\delta}$, summed over $\{\lambda \mid \lambda - \mu \text{ is a vertical } k\text{-strip}\}$. In any case, $\mathcal{E}_k P_\mu^{r\delta} = \sum_\lambda a_{\lambda\mu} P_\lambda^{r\delta}$ where λ is a partition of degree $|\mu| + k$. Evaluating at the point $x = \lambda + r\delta$ and using the expansion of $\mathcal{E}_k$, we see that $a_{\lambda\mu} P_\lambda^{r\delta}(\lambda + r\delta) = \mathcal{E}_k P_\mu^{r\delta}(x) = a_\delta(\lambda + r\delta)^{-1} \varphi_I(\lambda + r\delta) P_\mu^{r\delta}(\mu + r\delta)$. Hence, it remains to prove the identity

$$\psi'_{\lambda/\mu}(1/r) = a_\delta(\lambda + r\delta)^{-1} \varphi_I(\lambda + r\delta)(c_\lambda^{r\delta})^{-1} c_\mu^{r\delta}.$$

We first calculate $c_\lambda^{r\delta}/c_\mu^{r\delta} = r^{|\lambda|-|\mu|} c'_\lambda/c'_\mu$. Let us put $I' := \{i \notin I\}$, $J := \{\lambda_i \mid i \in I\}$ and $J' = \{\lambda_i \mid i \in I'\}$, and, for simplicity, let us write $c'_\lambda(i, j)$ instead of $c'_\lambda(1/r; (i, j))$. Then it is easy to see that for $i \in I$, we have $c'_\lambda(i, j + 1) = c'_\mu(i, j)$ unless $j \in J'$. Similarly, for $i \in I'$, $c'_\lambda(i, j) = c'_\mu(i, j)$ unless $j \in J$. Taking these cancellations into account, we get

$$\frac{c_\lambda^{r\delta}}{c_\mu^{r\delta}} = \frac{r^{|\lambda|} c'_\lambda}{r^{|\mu|} c'_\mu} = r^k \prod_{i \in I} c'_\lambda(i, 1) \prod_{i \in I, j \in J'} \frac{c'_\lambda(i, j + 1)}{c'_\mu(i, j)} \prod_{i \in I', j \in J} \frac{c'_\lambda(i, j)}{c'_\mu(i, j)}.$$

On the other hand, $a_\delta^{-1}(\lambda + r\delta)\varphi_I(\lambda + r\delta)$ equals

$$\prod_{i \in I}(\lambda_i + r\delta_i) \prod_{\substack{i \in I, k \in I' \\ i < k}} \frac{(\lambda_i + r\delta_i) - (\lambda_k + r\delta_k + r)}{(\lambda_i + r\delta_i) - (\lambda_k + r\delta_k)} \prod_{\substack{i \in I, k \in I' \\ k < i}} \frac{(\lambda_k + r\delta_k + r) - (\lambda_i + r\delta_i)}{(\lambda_k + r\delta_k) - (\lambda_i + r\delta_i)}.$$

Now the set $\{k \in I' \mid \lambda_k = 0\}$ equals $\{\lambda'_1 + 1, \lambda'_1 + 2, \ldots, n\}$, and for $j \in J'$, we have $\{k \in I' \mid \lambda_k = j\} = \{\lambda'_{j+1} + 1, \lambda'_{j+1} + 2, \ldots, \mu'_j\}$. Thus the first two products, which can be rewritten as

$$\prod_{i \in I}(\lambda_i + r(n - i)) \prod_{i \in I, k \in I', i < k} \frac{\lambda_i - \lambda_k + r(k - i - 1)}{\lambda_i - \lambda_k + r(k - i)},$$

become, after cancellation,

$$\prod_{i \in I}(\lambda_i + r(\lambda'_1 - i)) \prod_{\substack{i \in I, j \in J' \\ (i,j) \in \mu}} \frac{\lambda_i - j + r(\lambda'_{j+1} - i)}{\lambda_i - j + r(\mu'_j - i)} = r^k \prod_{i \in I} c'_\lambda(i, 1) \prod_{i \in I, j \in J'} \frac{c'_\lambda(i, j + 1)}{c'_\mu(i, j)}.$$

Finally, for each $j \in J$, the set $\{i \in I \mid \lambda_i = j\}$ equals $\{\mu'_j + 1, \mu'_j + 2, \ldots, \lambda'_j\}$. Thus, after cancellation, the third product $\prod_{j\in J,k\in I',k<i}\big(\lambda_k-\lambda_i+r(i-k+1)\big)\ \big(\lambda_k-\lambda_i+r(i-k)\big)$ becomes

$$\prod_{\substack{j\in J,k\in I'\\(k,j)\in\mu}}\frac{\lambda_k-j+r(\lambda'_j-k+1)}{\lambda_k-j+r(\mu'_j-k+1)}=\prod_{i\in I',j\in J}\frac{c_\lambda(i,j)}{c_\mu(i,j)}.$$

Since

$$\psi'_{\lambda/\mu}(1/r)=\prod_{i\in I',j\in J}\frac{c_\lambda(i,j)/c'_\lambda(i,j)}{c_\mu(i,j)/c'_\mu(i,j)},$$

the result follows. ■

7 Scholium

We close with a conjecture on the "integral" form of the Jack polynomial. In the homogeneous case, this is the function $J_\lambda^{(\alpha)}=c_\lambda(\alpha)P_\lambda^{(\alpha)}$. In the inhomogeneous situation, consider the function

$$J_\lambda^{r\delta}(x):=(-1)^{|\lambda|}c_\lambda(1/r)P_\lambda^{r\delta}(-x).$$

Various computations suggest the following extension of a conjecture of Macdonald for J_λ^{α}.

Conjecture. Put $\alpha=1/r$, and write $J_\lambda^{r\delta}=\sum_{\mu\le\lambda}\alpha^{|\mu|-|\lambda|}a_{\lambda\mu}(\alpha)m_\mu$. Then $a_{\lambda\mu}$ is a polynomial in α with positive integral coefficients. □

Recently, we have proved Macdonald's original conjecture as well as the integrality part of the above conjecture. We shall report on these developments elsewhere.

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