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Interpolation, Integrality, and a Generalization of Macdonald's Polynomials

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1 Introduction

A partition of length $\le n$ is a vector $\lambda \in \mathbb{Z}_+^n$ satisfying $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, and its weight is $|\lambda| = \lambda_1 + \cdots + \lambda_n$. The monomial symmetric function $m_\lambda(x)$ is the sum $\sum_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where α ranges over all distinct permutations of λ. The m_λ form a $\mathbb{Z}$-basis for the ring Λ_n of symmetric integral polynomials in $x_1, \ldots, x_n$.

Macdonald [M] has defined certain remarkable polynomials $P_\lambda(x; q, t)$ in $\Lambda_n \otimes \mathbb{Q}(q, t)$ which can be tersely characterized by the following two properties: First, in the expression of P_λ in terms of symmetric monomials, the coefficient of m_λ is 1. Second, let T_{q,x_i} be the "q-shift operator" defined by $T_{q,x_i} f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$; then P_λ is an eigenfunction with eigenvalue $\sum_{i=1}^n q^{\lambda_i} t^{n-i}$ for the operator D defined by

$$D := \sum_i A_i(x; t) T_{q,x_i} \quad \text{where } A_i(x; t) := \prod_{j \ne i} \frac{tx_i - x_j}{x_i - x_j}.$$

For $q = 0$ and $q = t$, one gets the *Hall-Littlewood* polynomial $P_\lambda(x; t)$ and the *Schur* polynomial s_λ, respectively, while $\lim_{t \to 1} P_\lambda(x; t^\alpha, t)$ yields the *Jack* polynomial $P_\lambda^{(\alpha)}(x)$.

Our first result is a generalization of $P_\lambda(x; q, t)$ to n "t-parameters." Thus let $\tau = (\tau_1, \ldots, \tau_n)$ be indeterminates, and put $\mathbb{F} = \mathbb{Q}(q, \tau)$. If μ is a partition, write $q^{-\mu}\tau$ for the n-tuple $(q^{-\mu_1}\tau_1, \ldots, q^{-\mu_n}\tau_n)$. We show that for each partition λ of length $\le n$ there is a unique (inhomogeneous) polynomial $R_\lambda(x; q, \tau)$ of degree $|\lambda|$ in $\Lambda_n \otimes \mathbb{F}$ which satisfies

(1) in the expansion of R_λ in terms of symmetric monomials, the coefficient of m_λ is 1;

(2) $R_\lambda(q^{-\mu}\tau; q, t) = 0$ for each partition $\mu \ne \lambda$ with $|\mu| \le |\lambda|$.

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1.1. Theorem. Let $R_\lambda(x;q,t)$ be the polynomial obtained from $R_\lambda(x;q,\tau)$ by specializing $\tau_i = t^{-(n-i)}$; then the top homogeneous component of $R_\lambda(x;q,t)$ is $P_\lambda(x;q,t)$. □

This is proved in Section 3, by showing that $R_\lambda(x;q,t)$ is an eigenfunction for a difference operator D' closely related to Macdonald's operator.

Our second result concerns a conjecture of Macdonald about $P_\lambda(x;q,t)$. We identify λ with its *diagram* consisting of the lattice points $(i,j) \in \mathbb{Z}^2$ such that $1 \le j \le \lambda_i$, and λ' denotes the transposed diagram. For $s = (i,j) \in \lambda$ the *armlength* is $a(s) = \lambda_i - j$ and the *leglength* is $l(s) = \lambda'_j - i$, and we put $c_\lambda(q,t) := \prod_{s\in\lambda}(1 - q^{a(s)}t^{l(s)+1})$.

The polynomial $J_\lambda(x;q,t) := c_\lambda(q,t)P_\lambda(x;q,t)$ has remarkable integrality properties. Write $m_d(\lambda)$ for the number of λ_i's equal to d. Let $S_\lambda(x;t)$ be the basis dual to s_λ for the inner product on $\Lambda_n \otimes \mathbb{Q}(t)$ defined by $\langle P_\lambda(x;t), P_\mu(x;t)\rangle = \delta_{\lambda\mu}/\prod_{d\ge1}\prod_{j=1}^{m_d(\lambda)}(1-t^j)$. Define the (q,t)-*Kostka* coefficients $K_{\lambda\mu}(q,t)$ by expressing

$$J_\mu(x;q,t) =: \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(x;t).$$

Our second main result, proved in Section 5, is the following.

1.2. Theorem. $K_{\lambda\mu}(q,t)$ is a polynomial in q and t with integral coefficients. □

Macdonald [M] (8.18?) has conjectured that the coefficients of $K_{\lambda\mu}(q,t)$ are actually positive integers, and Garsia and Haiman [GH] have even provided a conjectural representation-theoretic interpretation. Despite this, it was previously not even known that the $K_{\lambda\mu}(q,t)$ were polynomials.

Our proof of Theorem 1.2 actually raises more questions than it answers. In Section 4 we introduce a family of inhomogeneous nonsymmetric polynomials $G_\alpha(x;q,\tau)$ indexed by "compositions" $\alpha \in \mathbb{Z}_+^n$. These polynomials are closely connected with a remarkable representation of the Hecke algebra of the symmetric group, first defined by Bernstein and Zelevinsky. In Section 2 we collect relevant facts about the Hecke algebra and this representation.

Using operators from the Hecke algebra, we establish recursion formulas for the G_α and prove a polynomiality result for their coefficients. Now the R_λ can be obtained from the G_α by a symmetrization in the Hecke algebra, and hence we obtain Theorem 1.2.

We conclude with some remarks about the connection of our results with other work on the subject.

(a) For general τ the polynomials R_λ should probably be related to the q-Dyson identity (conjectured by G. Andrews, and proved in [BZ]) and to supersymmetric Schur functions [M, p. 90], but as yet we have not been able to make a precise connection.

(b) For $q = 0$, as proved by Lusztig [L1], the polynomials $K_{\lambda\mu}(q,t)$ are closely related to the Euler-Poincaré polynomials for the intersection cohomology sheaves on unipotent conjugacy classes in GL_m where $m = |\lambda| = |\mu|$.

(c) The representation of the Hecke algebra alluded to above can be generalized and extended to the "extended" affine Hecke algebra [L2] and even to the "double" affine Hecke algebra [C] for the Weyl group of an arbitrary root system. Macdonald's polynomials are also defined in this more general setting, and many of their properties have been established by Cherednik in this generality.

(d) Using our results, it is not too difficult to show that the top terms of G_α are eigenfunctions of Cherednik's operators [C] and hence are the "nonsymmetric" Macdonald polynomials for the case A_{n-1}. However, the deeper combinatorial properties of the A_{n-1} case do not seem to follow from the general case. Thus, despite their importance, in the interest of brevity we have omitted all discussion of Cherednik operators.

(e) The interpolation problems leading to R_λ and G_α had their genesis in the Capelli identity of [KS1], [KS2], [S], and in the classical (Jack polynomial) case have been studied in joint work with F. Knop, the details of which will appear elsewhere. In particular, in [KnS] we have settled in the affirmative a conjecture of Macdonald about the positivity and integrality of the coefficients of Jack polynomials.

Added in Proof. After this work was completed, we have learnt informally that Theorem 1.2 has recently been obtained independently by several people, including Garsia-Tesler, Garsia-Remmel, Knop, and Kirillov-Noumi. It should be instructive to compare their proofs with ours.

2 Interlude

In this section we review some facts about the symmetric group S_n and its Hecke algebra.

Let s_{ij} be the *transposition* in S_n which interchanges i and j. The set $\mathcal{S} = \{s_i := s_{i\,i+1}\}$ generates S_n, and we write $l(w)$ for the length of a *shortest*, or *reduced*, expression of w as a product of the s_i's. This length function satisfies $l(sw) = l(w) \pm 1$ for $s \in \mathcal{S}$.

Write $w > w'$ if $w = w's_{ij}$ for some transposition s_{ij} *and* if $l(w) > l(w')$. The transitive closure of this relation, still denoted by $>$, is called the *Bruhat order* on S_n.

2.1. Proposition. If $w' \geq w$ and $s \in \mathcal{S}$, then either $w's \geq w$ or $w's \geq ws$ (or both). □

(This is in [H, p. 119] with reversed inequalities. However, as explained on that page, the transformation $w \mapsto w_0 w$ yields the above form.)

A *composition* α of length $\leq n$ is simply a vector in $\mathbb{Z}_+^n$, and we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The symmetric group S_n acts on $\mathbb{Z}_+^n$, and the orbit of α contains a unique partition α^+.

Definition. For $\alpha \in \mathbb{Z}_+^n$ we define w_α in S_n by saying that for $i < j$, $w_\alpha(i) < w_\alpha(j)$ unless $\alpha_i < \alpha_j$.

2.2. Lemma. w_α is the unique minimal element among the $w \in S_n$ satisfying $w\alpha = \alpha^+$. □

Proof. Write $R := \{\epsilon_i - \epsilon_j \mid i \neq j\} \subset \mathbb{Z}^n$ where ϵ_i is the ith unit vector. Then R is a root system of type A_{n-1}, and $\Pi := \{\epsilon_i - \epsilon_j \mid i < j\}$ is the usual positive subsystem. A composition β is a partition if and only if for the usual inner product on $\mathbb{Z}^n$ we have $\langle \beta, \gamma \rangle \geq 0$ for all $\gamma \in \Pi$.

By [H, Ch. 1], the length of $w \in S_n$ is the cardinality of the set $\Pi(w) := \{\gamma \in \Pi \mid w(\gamma) \notin \Pi\}$, and w is uniquely determined by $\Pi(w)$. Moreover, it follows from the definition of w_α that $\Pi(w_\alpha)$ consists precisely of those γ in Π for which $\langle \alpha, \gamma \rangle < 0$.

Now if $w\alpha$ is a partition and $\gamma \in \Pi(w_\alpha)$, then we have $\langle w\alpha, w\gamma \rangle = \langle \alpha, \gamma \rangle < 0$, which implies that $w\gamma \notin \Pi$. Thus $\Pi(w) \supseteq \Pi(w_\alpha)$ and the result follows. ■

The *dominance* order for partitions (of length $\leq n$) is defined by writing $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for each $i < n$.

Definition. For α and β in $\mathbb{Z}_+^n$, we say that $\alpha \geq \beta$ if $|\alpha| = |\beta|$ and either

(1) $\alpha^+ > \beta^+$ in the dominance order, or

(2) $\alpha^+ = \beta^+$ and $w_\beta \geq w_\alpha$.

(Note the *reversed* Bruhat order inequality.)

2.3. Lemma. If $\alpha \geq \beta$ and $s \in \mathcal{S}$, then either $\alpha \geq s\beta$ or $s\alpha \geq s\beta$ (or both). □

Proof. If $\alpha^+ > \beta^+$ or if $\beta \geq s\beta$, then $\alpha \geq s\beta$. So assume $\alpha^+ = \beta^+$ and $s\beta > \beta$, which means that $l(w_{s\beta}) < l(w_\beta)$, and hence that $l(w_{s\beta}s) \leq l(w_\beta)$. Since $(w_{s\beta}s)\beta = \beta^+$ we also get $w_\beta \leq w_{s\beta}s$. This forces $l(w_\beta) = l(w_{s\beta}s)$ and hence $w_{s\beta} = w_\beta s$.

Now $\alpha \geq \beta$ implies $w_\beta \geq w_\alpha$, so by Proposition 2.1 either $w_{s\beta} \geq w_\alpha$ or $w_{s\beta} \geq w_\alpha s$. Since $(w_\alpha s)s\alpha = w_\alpha\alpha = \alpha^+$, we have $w_\alpha s \geq w_{s\alpha}$ and the result follows. ■

Definition. The *Hecke algebra* $\mathcal{H}$ of the symmetric group is the associative algebra over $\mathbb{Q}(t)$ generated by 1 and the elements T_s, $s \in \mathcal{S}$ subject to

(a) $T_s^2 = (1-t)T_s + t$;

(b) $T_{s_i}T_{s_j} = T_{s_j}T_{s_i}$ for $|i-j| > 1$;

(c) $T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$.

2.4. Proposition. If $w = s_{i_1} \cdots s_{i_k}$ is any reduced expression, then $T_{s_{i_1}} \cdots T_{s_{i_k}} =: T_w$ depends only on w. The elements T_w, $w \in S_n$ form a $\mathbb{Q}(t)$-basis for $\mathcal{H}$, and we have

(1) $T_v T_w = T_{vw}$ if $v \in S_n$ and $l(vw) = l(v) + l(w)$;

(2) $T_s T_w = (1-t)T_w + tT_{sw}$ if $s \in \mathcal{S}$ and $l(sw) < l(w)$. □

(This is proved in [H, Ch. 7] for the variant of $\mathcal{H}$ satisfying $T_s^2 = (q-1)T_s + qT_1$. However, this becomes (a) of our definition upon setting $q = t^{-1}$ and $T_s = t^{-1}T_s$.)

2.5. Lemma. The element $C = \sum_w t^{-l(w)} T_w$ satisfies $T_w C = C$ for all $w \in W$. □

Proof. It suffices to establish that $T_s C = C$ for s in $\mathcal{S}$. Partition $W = W_+ \coprod W_-$ according as $l(sw) - l(w)$ equals $+1$ or -1. Then $sW_\pm = W_\mp$, and we get $T_s C = \sum_+ t^{-l(w)} T_{sw} + (1 - t)\sum_- t^{-l(w)} T_w + t\sum_- t^{-l(w)} T_{sw} = t\sum_- + (1-t)\sum_- + \sum_+ = C$. ■

There is an important representation of $\mathcal{H}$, essentially due to Bernstein and Zelevinsky (unpublished). For $t = 0$ the representation was introduced by Bernstein, I. Gelfand, and S. Gelfand [BGG] and independently by Demazure [D].

Definition. Define $N_i := (x_i/(x_i - x_{i+1}))(1 - s_i)$ and $\sigma_i := s_i + (1-t)N_i$.

Observe that if f is a polynomial in $\{x_1, \ldots, x_n\}$, then $f - s_i f$ is divisible by $x_i - x_{i+1}$, and hence N_i and σ_i are well-defined operators on $\mathbb{Q}(t)[x_1, \ldots, x_n]$.

2.6. Proposition. $T_{s_i} \mapsto \sigma_i$ extends to a representation of $\mathcal{H}$ on $\mathbb{Q}(t)[x_1, \ldots, x_n]$. □

Proof. It is not too hard to verify (a), (b), and (c) directly. We sketch a proof and the interested reader can easily supply the details.

For (a), using $s_i^2 = 1$, $s_i x_i = x_{i+1} s_i$, and $s_i x_{i+1} = x_i s_i$, one checks that $N_i s_i + s_i N_i = s_i - 1$ and that $N_i^2 = N_i$. It follows that $\sigma_i^2 = (1-t)\sigma_i + t$.

Part (b) is trivial. For (c), one first checks that $\sigma_i x_i = x_{i+1}\sigma_i + (1-t)x_i$, $\sigma_i x_{i+1} = x_i \sigma_i - (1-t)x_i$, and $\sigma_i x_j = x_j \sigma_i$, if $j \neq i, i+1$. Now write $Z = \sigma_i\sigma_{i+1}\sigma_i - \sigma_{i+1}\sigma_i\sigma_{i+1}$. Using the above formulas, one may easily verify that $Zx_i = x_{i+2}Z$, $Zx_{i+2} = x_i Z$, and $Zx_j = x_j Z$ for $j \neq i, i+2$. Since $Z(1) = 0$, it follows by induction on $\deg(f)$ that $Z(f) = 0$ for all f, and hence that $Z = 0$. ■

For $\alpha \in \mathbb{Z}_+^n$ write $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

2.7. Proposition. (a) If $\alpha \leq \gamma$, then $\sigma_i x^\alpha$ is a combination of x^β with $\beta \leq \gamma$ or $\beta \leq s_i\gamma$.

(b) If $s_i\alpha > \alpha$, then the coefficient of $x^{s_i\alpha}$ in $\sigma_i x^\alpha$ is t. □

Proof. We have $\sigma_i x^\alpha = x^{s_i\alpha} + (1-t)N_i x^\alpha$ and

$$N_i x^\alpha = x_i \frac{x_i^{\alpha_i} x_{i+1}^{\alpha_{i+1}} - x_i^{\alpha_{i+1}} x_{i+1}^{\alpha_i}}{x_i - x_{i+1}} \prod_{j \neq i, i+1} x_j^{\alpha_j}.$$

Let l be the smaller of α_i, α_{i+1}, and put $k = |\alpha_i - \alpha_{i+1}|$. Then the monomials x^β which occur in $\sigma_i x^\alpha$ satisfy $\beta_j = \alpha_j$ for $j \neq i, i+1$, and $\beta_i = l + \epsilon$, $\beta_{i+1} = l + k - \epsilon$ for some

$0 \geq \epsilon \geq k$. Thus either $\beta = \alpha$ or $\beta^+ < \alpha^+$, or else $\beta = s_i\alpha$. Thus (a) follows, in the last case from Lemma 2.3.

For (b) we note that $s_i\alpha > \alpha$ means $\alpha_i < \alpha_{i+1}$, and so the coefficient of $x^{s_i\alpha}$ in $N_i x^\alpha$ is -1. Thus the coefficient in $\sigma_i x^\alpha = s_i x^\alpha + (1-t)N_i x^\alpha$ is $1 + (1-t)(-1) = t$. ■

3 Symmetric interpolation

Let q, k, and $\tau = (\tau_1, \dots, \tau_n)$ be indeterminates, and put $\mathbb{F} = \mathbb{Q}(q, k, \tau)$. If λ is a partition, let us write $k + q^{-\lambda}\tau$ for the n-tuple $(k + q^{-\lambda_1}\tau_1, \dots, k + q^{-\lambda_n}\tau_n)$.

3.1. Theorem. Symmetric polynomials of degree $\leq d$ in $\Lambda_n \otimes \mathbb{F}$ are uniquely determined by prescribing their values on the points $k + q^{-\lambda}\tau$ as λ ranges over partitions with $|\lambda| \leq d$. □

Proof. Writing $p = \sum c_\lambda m_\lambda$, interpolation gives a *square* linear system for c_λ. It suffices to prove existence, for which we may assume $n \geq 1$ and proceed by induction on $n + d$.

Now a partition λ of length $\leq n-1$ can be regarded as one of length $\leq n$ by appending a zero. Thus we get a natural, degree-preserving $\mathbb{Z}$-map $f \mapsto f^+$ from Λ_{n-1} to Λ_n extending $m_\lambda(x_1, \dots, x_{n-1}) \mapsto m_\lambda(x_1, \dots, x_n)$. This satisfies $f^+(x_1, \dots, x_{n-1}, 0) = f(x_1, \dots, x_{n-1})$.

We describe how to choose suitable symmetric polynomials f and g such that

$$p(x) := f^+(x_1 - k - \tau_n, \dots, x_n - k - \tau_n) + \left[\prod_{i=1}^{n}(x_i - k - \tau_n)\right] g(qx_1, \dots, qx_n)$$

has degree d and assumes prescribed values for $x = k + q^{-\lambda}\tau$ with $|\lambda| \leq d$.

First consider $x = k + q^{-\lambda}\tau$, as λ ranges over partitions with $|\lambda| \leq d$ and $\lambda_n = 0$. Then $x_n - k - \tau_n = 0$ and so the second term vanishes. The first term equals $f(x_1 - k - \tau_n, \dots, x_{n-1} - k - \tau_n)$ and its argument ranges over the set $(-\tau_n + q^{-\lambda_1}\tau_1, \dots, -\tau_n + q^{-\lambda_{n-1}}\tau_{n-1})$. By induction this determines f in $n-1$ variables with degree $\leq d$.

Now consider the points $x = k + q^{-\lambda}\tau$, as λ ranges over partitions such that $|\lambda| \leq d$ and $\lambda_n > 0$. If $d < n$, there are no such points and we put $g \equiv 0$. Otherwise $\mu = (\lambda_1 - 1, \dots, \lambda_n - 1)$ ranges over all partitions with $|\mu| \leq d - n$, and $(qx_1, \dots, qx_n) = qk + q^{-\mu}\tau$. Since each of the factors $x_i - k - \tau_n = (q^{-\lambda_i}\tau_i - \tau_n)$ is nonzero, by induction, we can find g with degree $\leq d - n$ such that $p(x)$ has the desired values at the remaining points. ■

3.2. Theorem. For each partition λ of length $\leq n$ there is a unique *inhomogeneous* polynomial $R_\lambda(x; k, q, \tau)$ of degree $|\lambda|$ in $\Lambda_n \otimes \mathbb{F}$ which satisfies

(1) in the expansion of R_λ in terms of symmetric monomials, the coefficient of m_λ is 1;

(2) $R_\lambda(k + q^{-\mu}\tau) = 0$ for each partition $\mu \neq \lambda$ with $|\mu| \leq |\lambda|$. □

Proof. By Theorem 3.1, the space of symmetric polynomials satisfying (2) is 1-dimensional (over $\mathbb{F}$). We need to show that the coefficient of m_λ for such polynomials is not *identically* zero. For this we examine the proof of Theorem 3.1, proceeding by induction on $n + |\lambda|$.

First suppose $\lambda_n > 0$, put $\mu = (\lambda_1 - 1, \dots, \lambda_n - 1)$, and put $g(x) = q^{-|\mu|} R_\mu(x; q, kq, \tau)$. Then $R_\lambda := \prod_{i=1}^n (x_i - k - \tau_n) g(qx_1, \dots, qx_n)$ satisfies both (1) and (2).

If $\lambda_n = 0$, write $\lambda_- = (\lambda_1, \dots, \lambda_{n-1})$, $\tau_- = (\tau_1, \dots, \tau_{n-1})$, and $x_- = (x_1, \dots, x_{n-1})$, and put $f(x_-) := R_{\lambda_-}(x_-; q, -\tau_n, \tau_-)$. By Theorem 3.1, for suitable g, the function $R_\lambda := f^+(x - k - \tau_n) + \prod_{i=1}^n (x_i - k - \tau_n) g(qx_1, \dots, qx_n)$ satisfies (2). The coefficient of m_λ is zero in the second term and, by induction and the definition of f^+, 1 in the first term. ■

It follows from the definitions that $\{R_\lambda\}$ is a basis (over $\mathbb{Q}(q, k, \tau)$) of $\Lambda_n \otimes \mathbb{Q}(q, k, \tau)$.

The dependence of R_λ on k is rather mild—it follows from the definition that $R_\lambda(x; q, k, \tau) = R_\lambda(x_1 - k, \dots, x_n - k; q, 0, \tau)$. Also, the only "divisions" involved in the construction of R_λ are by expressions of the form $(q^{-m}\tau_i - \tau_j)$ where $m > 0$ and $i \le j$. Thus we may specialize τ in any way we like, provided these expressions do not vanish. In particular, let $\delta = (n-1, \dots, 1, 0)$, and write $t^{-\delta}$ for the n-tuple $(t^{-(n-1)}, \dots, t^{-1}, 1)$.

Definition. We define $R_\lambda(x; q, t) := R_\lambda(x; q, 0, t^{-\delta})$.

We will show that the top homogeneous component of $R_\lambda(x; q, t)$ is the Macdonald polynomial $P_\lambda(x; q, t)$. The following operator plays a key role in the proof of this result.

Definition. We define

$$D' := \sum_i A_i(x; t)(1 - x_i^{-1})(1 - T_{q,x_i}), \quad \text{where } A_i(x; t) := \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j}.$$

3.3. Lemma. We have (a) $\sum_i A_i(x; t) = \sum_i t^{n-i}$, and (b) $\sum_i x_i^{-1} A_i(x, t) = \sum_i x_i^{-1}$. □

Proof. Let $a_\delta(x) := \prod_{i<j}(x_i - x_j) = \sum_{w \in S_n} (-1)^w x^{w\delta}$ be the Vandermonde determinant, and observe that we may rewrite $A_i(x; t) = a_\delta^{-1}(T_{t,x_i} a_\delta)$. For part (a) we have $\sum T_{t,x_i} a_\delta = \sum_i \sum_{w \in S_n} (-1)^w t^{(w\delta)_i} x^{w\delta}$, and the result follows by interchanging the order of summation.

For each $j > 1$, the coefficient of t^j in $\sum x_i^{-1} T_{t,x_i} a_\delta$ is a skew-symmetric *polynomial* of degree $< \deg(a_\delta)$ and so equals zero. Part (b) follows by setting $t = 1$. ■

3.4. Lemma. If f is a polynomial of degree $\le d$ in $\Lambda_n \otimes \mathbb{Q}(q, t)$, so is $D'f$. □

Proof. Observe that $(1 - T_{q,x_i})x^\lambda = (1 - q^{\lambda_i})x^\lambda$ and that this is zero if $\lambda_i = 0$, and thus $x_i^{-1}(1 - T_{q,x_i})$ maps polynomials to polynomials. If f is a symmetric polynomial, then $\sum_i (T_{t,x_i} a_\delta)(1 - x_i^{-1})(1 - T_{q,x_i}) f$ is skew-symmetric, and dividing by a_δ we conclude that $D'f$ is a symmetric polynomial of degree at most $\deg(f)$. ■

3.5. Lemma. $R_\lambda(x;q,t)$ is an eigenfunction of D' with eigenvalue $\sum t^{n-i} - \sum q^{\lambda_i} t^{n-i}$. □

Proof. First observe that $D'R_\lambda$ has degree $\le |\lambda|$. Next, put $x = q^{-\mu}t^{-\delta}$. Then $T_{q,x_i}f(x) = f(q^{-(\mu-\epsilon_i)}t^{-\delta})$, where ϵ_i is the ith unit vector in $\mathbb{Z}_+^n$.

If μ is a partition, so is $\mu - \epsilon_i$, unless either $\mu_i = \mu_{i+1} = m$ or $i = n, \mu_n = 0$. In the first case we have $x_i = q^{-m}t^{-n+i}$ and $x_{i+1} = q^{-m}t^{-n+i+1}$ thus $tx_i = x_{i+1}$ and so $A_i(x;t)$ vanishes. In the second case $x_n = 1$ and so $(1 - x_i^{-1})$ vanishes.

Combining these facts, we deduce that for $x = q^{-\mu}t^{-\delta}$ with $|\mu| \le |\lambda|$,

$$D'R_\lambda(x) = \begin{cases} 0 & \text{if } \mu \neq \lambda; \\ \sum A_i(x;t)(1 - x_i^{-1})R_\lambda(x) & \text{if } \mu = \lambda. \end{cases}$$

It follows that R_λ is an eigenfunction of D' with eigenvalue $\sum A_i(x;t)(1-x_i^{-1})$ where $x = q^{-\lambda}t^{-\delta}$. Lemma 3.3 completes the proof. ■

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Let us write P for the top homogeneous component of R_λ, and then the coefficient of m_λ in P is 1. Now the top term of $D'R_\lambda$ is $\sum A_i(x;t)(1 - T_{q,x_i})P = (\sum_i t^{n-i} - D)P$ where D is Macdonald's operator. Using Lemma 3.5 we conclude that $DP = \sum q^{\lambda_i}t^{n-i}P$. ■

4 Nonsymmetric interpolation

As before, let q, k, and $\tau = (\tau_1, \dots, \tau_n)$ be indeterminates and put $\mathbb{F} = \mathbb{Q}(q,k,\tau)$. The nonsymmetric case involves a "twist" by the permutation w_α defined in Section 2.

Definition. For $\alpha \in \mathbb{Z}_+^n$, we set $\overline{\alpha} = \overline{\alpha}(k,q,\tau) := k + q^{-\alpha}(w_\alpha\tau)$; i.e., $\overline{\alpha}_i = k + q^{-\alpha_i}(w_\alpha\tau)_i$.

We study the interaction of this definition with the following operations on n-tuples.

Definition. If $\eta = (\eta_1, \dots, \eta_n)$ is an n-tuple, we define

(1) $\eta_- := (\eta_1, \dots, \eta_{n-1})$;

(2) $\Phi_-\eta := (\eta_n - 1, \eta_1, \dots, \eta_{n-1})$, $\Phi_+\eta := (\eta_2, \dots, \eta_n, \eta_1 + 1)$;

(3) $\Phi_{q,k}\eta := (k(1-q) + q\eta_n, \eta_1, \dots, \eta_{n-1})$.

4.1. Lemma. For α in $\mathbb{Z}_+^n$,

(a) if $\alpha_n = 0$ then $\alpha_- \in \mathbb{Z}_+^{n-1}$ and $\overline{\alpha}_- = \overline{\alpha_-}(q,k,\tau_-)$;

(b) if $\alpha_n > 0$ then $\Phi_-\alpha \in \mathbb{Z}_+^n$ and $\Phi_{q,k}\overline{\alpha} = \overline{\Phi_-\alpha}$. □

Proof. If $\alpha_n = 0$, then from the definition of w_α it is clear that $w_\alpha(n) = n$ and that $w_\alpha(i) = w_{\alpha_-}(i)$ for $i < n$, which implies part (a). Next suppose $\alpha_n > 0$, and put $\beta = \Phi_-(\alpha)$, and then $\Phi_{q,k}\overline{\alpha} = (k + q^{-\beta_1}(w_\alpha\tau)_n, k + q^{-\beta_2}(w_\alpha\tau)_1, \dots, k + q^{-\beta_n}(w_\alpha\tau)_{n-1})$.

For $i < j < n$, $\alpha_i < \alpha_j \equiv \beta_{i+1} < \beta_{j+1}$, and thus $w_\alpha(i) < w_\alpha(j) \equiv w_\beta(i+1) < w_\beta(j+1)$; and for $j = n$, $\alpha_i < \alpha_n \equiv \beta_{i+1} \leq \beta_1$ so $w_\alpha(i) < w_\alpha(n) \equiv w_\beta(1) < w_\beta(i+1)$. This means $w_\beta\tau = ((w_\alpha\tau)_n, (w_\alpha\tau)_1, \dots, (w_\alpha\tau)_{n-1})$, and part (b) follows. ■

We can now prove the nonsymmetric analogues of Theorems 3.1 and 3.2.

4.2. Theorem. Polynomials of degree $\leq d$ in $\mathbb{F}[x_1, \dots, x_n]$ are uniquely determined by prescribing their values on the points $\overline{\alpha}$ for α in $\mathbb{Z}_+^n$ with $|\alpha| \leq d$. □

Proof. As before, the interpolation problem is a *square* linear system and existence implies uniqueness. For existence, we argue by induction on $n + d$ and may assume $n \geq 1$.

We will find suitable polynomials f and g so that

$$p(x) := f(x_1 - k - \tau_n, \dots, x_{n-1} - k - \tau_n) + (x_n - k - \tau_n)g(\Phi_{q,k}x)$$

has prescribed values for $x = \overline{\alpha}$ with $|\alpha| \leq d$.

First consider the points $x = \overline{\alpha}$ as α ranges over $\mathbb{Z}_+^n$ with $|\alpha| \leq d$ and $\alpha_n = 0$. By Lemma 4.1 (a), we see that $x_n = k + \tau_n$ so that the second term vanishes, and also that the argument of f ranges over $\overline{\alpha_-}(q, -\tau_n, \tau_-)$ for α_- in $\mathbb{Z}_+^{n-1}$ with $|\alpha_-| \leq d$. By induction this determines f in $n - 1$ variables with degree $\leq d$.

Now consider the points $x = \overline{\alpha}$, for $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq d$ and $\alpha_n = l > 0$. Then $x_n - k - \tau_n = q^{-l}\tau_i - \tau_n$ for some i, and this is nonzero. By Lemma 4.1 (b), the argument of g ranges over the points $\overline{\beta}$ for $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq d - 1$. By induction, we can find g with degree $\leq d - 1$ such that $p(x)$ has the desired values at the remaining points. ■

4.3. Theorem. For each α in $\mathbb{Z}_+^n$ there exists a unique *inhomogeneous* polynomial $G_\alpha := G_\alpha(x; q, k, \tau)$ of degree $\leq |\alpha|$ in $\mathbb{F}[x_1, \dots, x_n]$ which satisfies

(1) the coefficient of x^α in G_α is 1;

(2) $G_\alpha(\overline{\beta}) = 0$ for each $\beta \neq \alpha$ in $\mathbb{Z}_+^n$ with $|\beta| \leq |\alpha|$. □

Proof. As in the symmetric case, the uniqueness is clear and for existence we examine the proof of Theorem 4.2, proceeding by induction on $n + |\alpha|$.

If $\alpha_n > 0$, then let $\beta = \Phi_-\alpha := (\alpha_n - 1, \alpha_1, \dots, \alpha_{n-1})$, and put $g(x) = q^{-\alpha_n+1}G_\beta(x)$. Then $G_\alpha := (x_n - k - \tau_n)g(\Phi_{q,k}x)$ satisfies both (1) and (2).

If $\alpha_n = 0$, then write $f := G_{\alpha_-}(x_-; q, -\tau_n, \tau_-)$. By Theorem 4.2, for suitable g, the function $G_\alpha := f(x) + (x_n - k - \tau_n)g(\Phi_{q,k}x)$ satisfies (2). The coefficient of x^α is zero in the second term and, by induction, it is 1 in the first term. ■

As before, the specialization $\tau = t^{-\delta}$ is well defined and leads to remarkable functions.

Definition. We define $G_\alpha(x; q, t) := G_\alpha(x; q, 0, t^{-\delta})$.

The first basic property of these functions has already been established in the proof of Theorem 4.3. For ease of future reference we formulate it as a corollary.

4.4. Corollary. Let Φ_q be the operator $\Phi_q f(x) := (x_n - 1) f(qx_n, x_1, \ldots, x_{n-1})$, suppose $\alpha_n > 0$, and put $\beta = (\alpha_n - 1, \alpha_1, \ldots, \alpha_{n-1})$. Then $G_\alpha(x; q, t) = q^{-\alpha_n+1} \Phi_q G_\beta(x; q, t)$. □

Recall from Section 2 that

$$\sigma_i := s_i + (1-t)\frac{x_i}{x_i - x_{i+1}}(1 - s_i)$$

generate a representation of the Hecke algebra $\mathcal{H}$ on $\mathbb{Q}(t)[x_1, \ldots, x_n]$. (Hence also on $\mathbb{Q}(q, t)[x_1, \ldots, x_n]$.)

4.5. Theorem. Write G_α for $G_\alpha(x; q, t)$.

(a) If $s_i\alpha = \alpha$, then $\sigma_i G_\alpha = G_\alpha$ and $s_i G_\alpha = G_\alpha$.

(b) If $s_i\alpha \neq \alpha$, then $[(1 - \overline{\alpha}_{i+1}/\overline{\alpha}_i)\sigma_i + t - 1]G_\alpha$ is a nonzero multiple of $G_{s_i\alpha}$. □

Proof. First note that we can rewrite

$$\sigma_i G_\alpha(x) = \frac{x_i - tx_i}{x_i - x_{i+1}} G_\alpha(x) + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}} G_\alpha(s_i x). \tag{$*$}$$

Next we make two crucial observations. First, if $\beta_i \neq \beta_{i+1}$, then $w_{s_i\beta} = s_i w_\beta$ and hence $s_i\overline{\beta} = \overline{s_i\beta}$. Second, if $\beta_i = \beta_{i+1} = b$ and $w_\beta(i) = j$, say, then $w_\beta(i+1) = j+1$ and so $t\overline{\beta}_i = tq^{-b}t^{-(n-j)} = q^{-b}t^{-(n-(j+1))} = \overline{\beta}_{i+1}$.

Now suppose $s_i\alpha = \alpha$ and β in $\mathbb{Z}_+^n$ satisfies $|\beta| \leq |\alpha|$ and $\beta \neq \alpha$. Then $s_i\beta \neq \alpha$, and if we substitute $x = \overline{\beta}$ in ($*$), then by the above remarks, we see that both terms vanish. Also, if we substitute $x = \overline{\alpha}$, the second term vanishes while the first becomes $G_\alpha(\overline{\alpha})$. Since $\deg(\sigma_i G_\alpha) = \deg(G_\alpha)$, Theorem 4.2 implies that $\sigma_i G_\alpha = G_\alpha$. The implication $s_i G_\alpha = G_\alpha$ is a formal consequence. Indeed, if f is any function such that $\sigma_i f = f$, then we get

$$0 = \sigma_i f - f = \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(s_i f - f)$$

and hence $s_i f - f = 0$. This proves (a).

Finally, suppose $s_i\alpha \neq \alpha$ and $|\beta \leq |\alpha|$, and then by the above remarks we get

$$\sigma_i G_\alpha(\overline{\beta}) = \begin{cases} 0 & \text{if } \beta \neq \alpha, s_i\alpha; \\ \dfrac{\overline{\alpha}_i - t\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}} G_\alpha(\overline{\alpha}) & \text{if } \beta = \alpha; \\ \dfrac{t\overline{\alpha}_{i+1} - \overline{\alpha}_i}{\overline{\alpha}_{i+1} - \overline{\alpha}_i} G_\alpha(\overline{\alpha}) & \text{if } \beta = s_i\alpha. \end{cases}$$

Rewriting this, we get

$$[(\overline{\alpha}_i - \overline{\alpha}_{i+1})\sigma_i + (t-1)\overline{\alpha}_i]G_\alpha(\overline{\beta}) = \begin{cases} 0 & \text{if } \beta \neq s_i\alpha; \\ [\overline{\alpha}_i - t\overline{\alpha}_{i+1}]G_\alpha(\overline{\alpha}) & \text{if } \beta = s_i\alpha. \end{cases}$$

Since $\alpha_i \neq \alpha_{i+1}$, we get $\overline{\alpha}_i - t\overline{\alpha}_{i+1} \neq 0$, and part (b) follows from Theorem 4.2. ∎

4.6. Theorem. For a partition λ, let V_λ be the $\mathbb{Q}(q,t)$-span of $\{G_{w\lambda}(x;q,t) \mid w \in S_n\}$. Then V_λ is $\mathcal{H}$-invariant, and for each $f \in V_\lambda$, $\sum t^{-l(w)}T_w f$ is proportional to $R_\lambda(x;q,t)$. □

Proof. Since the σ_i generate $\mathcal{H}$, part (a) follows immediately from Theorem 4.5.

For part (b), put $R = \sum t^{-l(w)}T_w f$. By Lemma 2.5, $\sigma_i R = R$ for each i, and hence, as observed in the proof of Theorem 4.5, $s_i R = R$, and so R is a symmetric polynomial. Since $\deg(R) \le |\lambda|$, by Theorem 3.2 it suffices to show that if μ is a partition with $|\mu| \le |\lambda|$ and $\mu \neq \lambda$, then $R(q^{-\mu}t^{-\delta}) = 0$. But in this case $w_\mu = 1$ and $\overline{\mu} = q^{-\mu}t^{-\delta}$, and so every $G_{w\lambda}$ vanishes at $q^{-\mu}t^{-\delta}$. Thus so does every function in V_λ, including R. ∎

5 Integrality

We now undertake a detailed study of the coefficients of R_λ and G_α with respect to the monomial bases. This culminates in the proof of Theorem 1.2.

Recall the partial order $\ge$ defined in Section 2 for the set of α in $\mathbb{Z}_+^n$ of a fixed weight. We now extend this, still writing $\ge$, by including the relations $\alpha \ge \beta$ if $|\alpha| > |\beta|$.

5.1. Theorem. The coefficient of x^β in $G_\alpha(x;q,t)$ is zero unless $\beta \le \alpha$. □

Proof. The case $|\alpha| = 0$ is trivial, and we proceed by induction on $|\alpha|$. Now if $s_i\alpha \ge \alpha$ and if the result holds for α, then by Lemma 2.3 and Theorem 4.5 it also holds for $s_i\alpha$.

Thus we may assume that α is *antidominant*, i.e., satisfies $\alpha_1 \le \cdots \le \alpha_n$. We need to show that if x^β with $\beta \neq \alpha$ occurs in G_α, then either $\beta^+ < \alpha^+$ or $|\beta| < |\alpha|$.

Let k be the smallest index such that $\alpha_k = \alpha_{k+1} = \cdots = \alpha_n = a$, say, and put $\gamma = (\Phi_-)^{n-k+1}\alpha = (a-1,\dots,a-1,\alpha_1,\dots,\alpha_{k-1})$. Then, by Corollary 4.4, G_α is proportional to $\Phi_q^{n-k+1}G_\gamma$. Thus, if x^β occurs in G_α, then either $|\beta| < |\alpha|$ (in which case we are done), or else there is some $\eta < \gamma$ with $|\eta| = |\gamma|$ and $\beta = \Phi_+^{n-k+1}\eta$.

Since $\gamma^+ \ge \eta^+$ and since each coordinate of γ is $\le a-1$, either we have $\alpha^+ > \eta^+$ (in which case we are done), or else the first $n-k+1$ coordinates of η are also $a-1$. In the latter case, since the last $k-1$ coordinates of γ form an antidominant tuple, we have either $\gamma^+ > \eta^+$, which implies $\alpha^+ > \beta^+$, or $\gamma = \eta$, which gives $\alpha = \beta$. ∎

For $i < j$, s_{ij} has the reduced expression $s_i s_{i+1}\cdots s_{j-2}s_{j-1}s_{j-2}\cdots s_i$ (and length $2j-2i-1$). Hence, in the Hecke algebra, $T_{ij} := T_{s_{ij}} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i$.

We now prove a refinement of Theorem 4.5 for s_{ij} in the following setting: Suppose $\alpha \in \mathbb{Z}_+^n$ satisfies $\alpha_k = 0$ for some k, and $\alpha_n \neq 0$. Let j be the *largest* integer with $\alpha_{j-1} = 0$, and let i be *any* integer such that $\alpha_i = \alpha_{i+1} = \cdots = \alpha_{j-1} = 0$, and put $\beta = s_{ij}\alpha$.

5.2. Theorem. Let α, β be as above with $\overline{\alpha}_j = q^{-a}t^{-d}$, and put $e = d + i - j + 1$; then

$$t^{j-i}(1 - q^{-a}t^{-e})G_\beta = (1 - q^{-a}t^{-e})T_{ij}G_\alpha + (t-1)\sum_{k=i+1}^{j} t^{k-i-1}T_{kj}G_\alpha. \qquad \square$$

Proof. For $i \le k < j$, $\alpha_k = 0$, and so $\overline{\alpha}_k = t^{k-j+1}$, and $\beta = s_i s_{i+1} \cdots s_{j-1}\alpha$. Writing $\{k\} = 1 - \overline{\alpha}_j/\overline{\alpha}_k = 1 - q^{-a}t^{-d-k+j-1}$ and $[k] = \{k\}\sigma_k + t - 1$, by Theorem 4.5 we get

$$G_\beta \sim [i][i+1]\cdots[j-1]G_\alpha.$$

(The notation $\sim$ means "is a nonzero multiple of.")

Now $\{k\} + t - 1 = 1 - q^{-a}t^{-d-k+j-1} + t - 1 = t\{k+1\}$. Also, for $i \le k \le j-2$, by Theorem 4.5, $\sigma_k G_\alpha = G_\alpha$ and so $[k]G_\alpha = t\{k+1\}G_\alpha$. Expanding $[j-1]G_\alpha$, and using this, we observe that we can *pull out a factor of* $\{j-1\}$ to get

$$G_\beta \sim [i][i+1]\cdots[j-2]\sigma_{j-1}G_\alpha + (t-1)t^{j-i-1}\{i+1\}\cdots\{j-2\}G_\alpha.$$

For $k < j-2$, σ_k and σ_{j-1} commute, and so $\sigma_{j-1}G_\alpha$ is invariant under σ_k. Thus, expanding $[j-2]$ and arguing as above, we get a three-term expression in which the factor $\{j-2\}$ can be pulled out. Continuing in this manner, we conclude that

$$G_\beta \sim \{i\}\sigma_i\cdots\sigma_{j-1}G_\alpha + (t-1)\sum_{k=i+1}^{j} t^{k-i-1}\sigma_k\cdots\sigma_{j-1}G_\alpha.$$

Since $\{i\} = 1 - q^{-a}t^{-e}$ and since σ_k fixes G_α for $i \le k < j-1$, we can rewrite this as

$$G_\beta \sim (1 - q^{-a}t^{-e})T_{ij}G_\alpha + (t-1)\sum_{k=i+1}^{j} t^{k-i-1}T_{kj}G_\alpha.$$

It remains to show that the coefficient of x^β on the right side is $t^{j-i}(1 - q^{-a}t^{-e})$. By Theorem 5.1 and Proposition 2.7 (a), the only part of the expression in which x^β occurs is $(1 - q^{-a}t^{-e})\sigma_i\cdots\sigma_{j-1}x^\alpha$. The result follows from Proposition 2.7 (b). ■

The previous formulas allow us to control the coefficients of G_α. The sharpest results are obtained when α is *antidominant*. Thus, let λ be a partition and let $\alpha = (\lambda_n, \ldots, \lambda_1)$.

5.3. Theorem. The coefficients of $c_\lambda(q^{-1}, t^{-1})G_\alpha$ are polynomials in $\mathbb{Z}[q^{-1}, t, t^{-1}]$. $\square$

Proof. This is obvious if $|\lambda| = 0$, and we proceed by induction on $|\lambda|$. Let l be the length of λ, write $\mu = (\lambda_1 - 1, \dots, \lambda_l - 1, 0, \dots, 0)$, and put

$$\gamma = (\mu_n, \dots, \mu_1) = (0, \dots, 0, \lambda_l - 1, \dots, \lambda_1 - 1),$$
$$\eta = (\lambda_l - 1, \dots, \lambda_1 - 1, 0, \dots, 0).$$

By induction, $c_\mu(q^{-1}, t^{-1})G_\gamma$ has coefficients in $\mathbb{Z}[q^{-1}, t, t^{-1}]$, and we consider how these change as we go from G_γ to G_η and then to G_α.

Now we can transform γ to η as follows: Let γ_h be the first nonzero entry of γ. For each $j = h, \dots, n$, successively apply the transpositions s_{ij} for $i = j - (n - l)$ to exchange $\gamma_j = \lambda_{n-j+1} - 1$ with the zero entry $n - l$ places above it.

Theorem 5.2 applies to this situation, with $a = \lambda_{n-j+1} - 1$ and $e = n - w_\gamma(j) + (j - n + l) - j + 1 = l - w_\gamma(j) + 1$. It follows that as j ranges from h to n, the pairs $(a, e - 1)$ range over the arm- and leglengths of the lattice points $(k, 1) \in \lambda$ for those k with λ_k at least 2. Thus $1 - q^{-a}t^{-e}$ ranges over the (q^{-1}, t^{-1})-hooklengths for these lattice points. Throwing in terms of the form $(1 - t^{-m})$ for the remaining hooklengths, we deduce from Theorem 5.2 that the coefficients of $c_\lambda(q^{-1}, t^{-1})G_\eta$ are in $\mathbb{Z}[q^{-1}, t, t^{-1}]$.

Next, by repeated applications of Corollary 4.4, we have

$$G_\alpha = q^{-|\eta|} \prod_{i=n-l+1}^{n} (x_i - 1) G_\eta(qx_{n-l+1}, \dots, qx_n, x_1, \dots, x_{n-l}).$$

Now if x^β occurs in G_η, then, by Theorem 5.1, we have $|\beta| \le |\eta|$, and it follows that every coefficient of $c_\lambda(q^{-1}, t^{-1})G_\alpha$ is in $\mathbb{Z}[q^{-1}, t, t^{-1}]$. ■

We now prove the symmetric version of the previous result.

5.4. Theorem. The coefficients of $c_\lambda(q^{-1}, t^{-1})R_\lambda(x; q, t)$ are polynomials in $\mathbb{Z}[q^{-1}, t, t^{-1}]$. □

Proof. For $\alpha = (\lambda_n, \dots, \lambda_1)$, by Theorem 4.6 we know that the sum $\sum_{w \in S_n} t^{-l(w)} T_w G_\alpha$ is proportional to R_λ. In fact, we can restrict the sum to a certain coset described below.

Thus let $I := \{s \in \mathcal{S} \mid s(\lambda) = \lambda\}$ (in other words, $s_i \in I$ if and only if $\lambda_i = \lambda_{i+1}$), and let W_I be the subgroup of S_n generated by I. Then, by [H, p. 19], there is a set W^I in S_n such that for every w in W there exist unique $u \in W^I$ and $v \in W_I$ such that $w = uv$. Moreover, we have $l(w) = l(u) + l(v)$, which implies that $T_w = T_u T_v$.

Now by Theorem 4.5, $T_v F_\alpha = F_\alpha$, thus pulling out a factor of $\sum t^{-l(v)}$, we conclude that $\sum_{u \in W^I} t^{-l(u)} T_u G_\alpha$ is proportional to R_λ. Now, by Theorem 5.1, the only term which contains x^λ corresponds to the unique element u_o such that $u_o \alpha = \lambda$, and by Proposition 2.7 (b) the coefficient of x^λ in $T_{u_o} x^\alpha$ is $t^{l(u_o)}$.

Thus R_λ is *equal* to $\sum_{u \in W^I} t^{-l(u)} T_u G_\alpha$. Since T_w preserves the space of polynomials with coefficients in $\mathbb{Z}[q^{-1}, t, t^{-1}]$, the theorem follows from Theorem 5.3. ■

Finally we prove that $K_{\lambda\mu}(q, t) \in \mathbb{Z}[q, t]$.

Proof of Theorem 1.2. Since by Theorem 1.1 P_λ is the top homogeneous component of R_λ, we conclude that for each λ the coefficients of $c_\lambda(q^{-1}, t^{-t}) P_\lambda(x; q, t)$ are in $\mathbb{Z}[q^{-1}, t^{-1}, t]$.

Now by [M, p. 324], $P_\lambda(x; q^{-1}, t^{-1}) = P_\lambda(x; q, t)$. Replacing q, t by their inverses, we deduce that the coefficients of $J_\lambda(x; q, t)$ are in $\mathbb{Z}[q, t, t^{-1}]$.

By [M, p. 364], for partitions of a fixed weight $k \geq 0$, the transition matrix from the $S_\lambda(x; t)$ basis to the m_λ basis has entries in $\mathbb{Z}[t]$ and its determinant is $\prod_{|\lambda|=k} c_\lambda(0, t)$, which is a product of terms of the form $1 - t^d$ for various integers $d > 0$.

Applying the inverse transition matrix and clearing denominators, we deduce that for each λ and μ there are polynomials $K'(q, t) \in \mathbb{Z}[q, t]$ and $K''(t) \in \mathbb{Z}[t]$ such that $K_{\lambda\mu}(q, t) = K'(q, t)/K''(t)$, and K'' is a product of terms of form t^a and $1 - t^d$. Thus the lowest coefficient of K'' is 1, and for $t \in \mathbb{C}$ with $|t| \neq 0, 1$, $K_{\lambda\mu}(q, t)$ is a polynomial in q.

However, by [M, p. 354], $K_{\lambda\mu}(q, t) = K_{\lambda'\mu'}(t, q)$ where, as usual, λ' and μ' denote the transposed partitions. Applying the previous remarks to $K_{\lambda'\mu'}(t, q)$, we conclude that it, and hence $K_{\lambda\mu}(q, t)$, is a polynomial in t for generic q. This means $K''(t)$ *divides* $K'(q, t)$. Since the lowest coefficient of K'' is 1, we conclude that $K_{\lambda\mu} \in \mathbb{Z}[q, t]$. ■

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