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LOGARITHMIC CONVEXITY OF PERRON-FROBENIUS EIGENVECTORS OF POSITIVE MATRICES

SIDDHARTHA SAHI

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ABSTRACT. Let $C(S)$ be the cone of Perron-Frobenius eigenvectors of stochastic matrices that dominate a fixed substochastic matrix S. For each $0 \le \alpha \le 1$, it is shown that if u and v are in $C(S)$ then so is w, where $w_j = u_j^\alpha v_j^{1-\alpha}$.

The basic result of Perron-Frobenius theory [S] is that if a matrix has strictly positive entries, then its maximal eigenvalue is unique, positive, occurs with multiplicity 1, and has a (coordinatewise) positive eigenvector.

Subsequent literature on positive matrices contains many results (e.g., [C, F, K]) that deal with convexity properties of the dominant *eigenvalue* as a function of matrix entries. Similar results for the corresponding *eigenvectors* are obtained in [DN, EJN] but *only* for the effects of varying a *single* row of the matrix. Little seems to be known about the behavior of these eigenvectors under a more general perturbation of the matrix.

In this paper we prove a different kind of convexity property for Perron-Frobenius eigenvectors that was motivated by economic considerations in [SY] but that, by virtue of its unexpected and elementary nature, seems to warrant a wider mathematical audience.

For convenience, we formulate the result in terms of stochastic matrices—a positive matrix is called *stochastic* (*substochastic*) if its column sums are equal to (less than) 1.

For a fixed substochastic matrix S, consider the cone $C(S)$ of all (positive) Perron-Frobenius eigenvectors of the various stochastic matrices that (entrywise) dominate S. Thus $C(S) = \{v > 0 \mid \exists \text{ stochastic } A \ge S \text{ such that } Av = v\}$.

Now $C(S)$ need not be a convex subset of $\mathbf{R}^n$. However, we shall show that it has the following remarkable property that may be termed logarithmic, or geometric, convexity.

Theorem. *Fix* $0 \le \alpha \le 1$, *and put* $\beta = 1 - \alpha$. *If* $u = (u_1, u_2, \dots, u_n)$ *and* $v = (v_1, v_2, \dots, v_n)$ *are in* $C(S)$ *then so is* $w = (w_1, w_2, \dots, w_n)$ *where* $w_j = u_j^\alpha v_j^\beta$.

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The principal difficulty in proving this theorem is the indirect nature of the definition of $\mathrm{C}(S)$. The following lemma "eliminates the quantifier" in that definition.

Lemma. *A positive vector* v *belongs to* $\mathrm{C}(S)$ *if and only if* $Sv \leq v$.

Proof. If v is in $\mathrm{C}(S)$, choose $A \geq S$ such that $v = Av$. Then clearly $Sv \leq Av = v$.

Conversely, suppose $v > 0$ with $Sv \leq v$, and put $\delta = v - Sv$. Also, let s_j be the jth column sum of S, and put $\varepsilon_j = 1 - s_j$. Clearly, $0 < \varepsilon_1 v_1 + \cdots + \varepsilon_n v_n = \delta_1 + \cdots + \delta_n = \lambda$, say. Now let A be the matrix whose ijth entry is $a_{ij} = s_{ij} + \frac{1}{\lambda}\delta_i \varepsilon_j$. It is easily checked that A is stochastic, dominates S, and satisfies $Av = v$. □

Proof of Theorem. In view of the lemma, we may assume that $Su \leq u$ and $Sv \leq v$, and we have to show that $Sw \leq w$. Using the Hölder inequality, we get

$$(Sw)_i = \sum_j s_{ij} w_j = \sum_j s_{ij} u_j^{\alpha} v_j^{\beta} = \sum_j (s_{ij} u_j)^{\alpha} (s_{ij} v_j)^{\beta}$$
$$\leq \left(\sum_j s_{ij} u_j\right)^{\alpha} \left(\sum_j s_{ij} v_j\right)^{\beta} \leq (u_i)^{\alpha} (v_i)^{\beta} = w_i. \quad \square$$

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Department of Mathematics, Princeton University, Princeton, New Jersey 08544
E-mail address: sahi@math.princeton.edu