

Repeated trade and the velocity of money*

Pradcep Dubey

State University of New York, Stony Brook NY, USA

Siddhartha Sahi

Princeton University, Princeton NJ, USA

Martin Shubik

Yale University, New Haven CT, USA

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There are two sources of inefficiency of strategic equilibria (SE) in market mechanisms. The first is the oligopolistic effect, which occurs when an agent can single-handedly influence prices. With a continuum of agents we get 'perfect competition' and this effect is, of course, wiped out. But the inefficiency of SEs may nevertheless persist because agents are not 'perfectly liquid', i.e., the constraints of the mechanism are such that they cannot carry out arbitrary trades at the market prices. Our main result is a *formal* proof of the fact that, if enough repeated rounds of trade are permitted within a single utility period, then the liquidity problem is overcome: SE outcomes turn out to be not only efficient but, in fact, Walrasian.

1. Introduction

There are two sources of inefficiency of strategic equilibria (SE) in market mechanisms. The first is the oligopolistic effect, which occurs when an agent can single-handedly influence prices. With a continuum of agents we get 'perfect competition' and this effect is, of course, wiped out. But the inefficiency of SE's may nevertheless persist because agents are not 'perfectly liquid,' i.e., the constraints of the mechanism are such that they cannot carry out arbitrary trades at the market prices. Our main result is that, if enough repeated rounds of trade are permitted within a single utility period, then the

Correspondence to: Siddhartha Sahi, Department of Mathematics, Princeton University, Princeton, NJ 08544, USA.

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liquidity problem is overcome: SE outcomes turn out to be not only efficient but, in fact, Walrasian.

A typical case arises when money is used as a medium of exchange, and the purchases by an agent must be financed out of the money he has on hand at the time. Here an insufficient supply of money, or its maldistribution, is often the cause of illiquidity [Dubey and Shapley (1976)]. The upshot of repeated trade is to bring about an increase in the velocity of money. This in turn restores liquidity to the economy, no matter how constraining the one-shot mechanism might have been. (See the examples in section 5.)

Our approach is similar to that of Dubey et al. (1980) but there are differences. As in Dubey et al. (1980), we start with an abstract market mechanism which maps agents' moves into trades. There are *no* convexity, anonymity, aggregation or continuity axioms imposed on the mechanism [all of which were needed in Dubey et al. (1980)]. Instead we suppose that agents' moves 'typically' produce prices which mediate trade. This is indeed a key *conclusion* of Dubey et al. (1980). But it seems to us quite reasonable to directly suppose this property because it holds for most mechanisms of interest to us that have been studied in the literature (see references).

In all of these mechanisms, trade is conducted through markets, and there is a natural notion of 'doing nothing' at one or more of these markets. Thus *certain* move-selections do *not* lead to the formation of *all* prices. While our framework is abstract, and in particular does not portulate the existence of markets, we do want it to include the above mechanisms. So we assume that not trading is an option that is always available to every agent. (This seems to us quite natural on its own). Moreover, we admit the possibility of move selections at which prices are undefined for some commodities (though mechanisms in which prices always form are, of course, not ruled out by us). But then trivial S.E.'s can exist (e.g. all traders do nothing!) which, most of the time, will be nowhere near Walrasian. In the examples cited, this is tantamount to the fact that many markets 'fail to form'. This scenario must be distinguished from a superficially similar one in which all markets are 'open for business' via announced prices but traders *choose* to do nothing. We achieve the distinction in our framework via the notion of an 'open' S.E. A closely related notion is that of an 'active' S.E., i.e., one in which move selections are such that all prices form in the mechanism, in each trading period. Obviously, every active S.E. is open.

The other key assumption we make is that, in a weak sense, the mechanism is 'connected.' This means that the mechanism does not a priori rule out the *ultimate* conversion of commodity i into j (over many rounds of trade for any pair i, j).

Given these assumptions and a continuum of agents, we show that all the open S.E.'s of the game with T -rounds of trade will be close to Walrasian, if T is large enough. Furthermore, with an additional nondegeneracy assump-

tion on the mechanism, all Walrasian outcomes will be achieved at active (hence open) SE's for all sufficiently large T . (Incidentally, this proves the existence of active S.E.'s!)

Let us relate this result to Dubey et al. (1980). It was shown in Dubey et al. (1980) that, in a one-period setting, all 'full' S.E.'s are Walrasian. 'Fullness' meant that each agent could obtain trades in a (codimension one) neighborhood of his final bundle (i.e., in our parlance, was fully liquid). The question as to when 'fullness' would obtain was left open in Dubey et al. (1980).

It is easy to see that 'openness' is a much weaker assumption than 'fullness' (for models which belong to both frameworks). The paper Dubey and Shapley (1976) for instance, deals with open S.E.'s that are not full, because agents end up spending 'all their money'. (See also the examples in section 5.)

Finally it is worth pointing out that our framework is general enough to include both Bertrand and Cournot competition (see, again, the examples in section 5). The fact that we do *not* impose continuity (of outcomes with respect to strategies) is crucial here; for continuity is bound to fail in any Bertrandian model, on account of 'price-cutting' behaviour.

2. The exchange economy

The set of agents I is the closed unit interval $[0, 1]$. The set of commodities is $I_m = \{1, \dots, m\}$. We will use Greek letters α, β for agents and Latin letters i, j for commodities. Depending on the context, vectors in the non-negative orthant $\mathbb{R}_+^m \equiv \mathbb{R}_+^m$ will represent commodity bundles or prices.

The *economy* E is a pair (e, u) where $e: I \rightarrow \mathbb{R}_+^m$ is bounded and measurable with $\int e \gg 0$ [where $\int h$ denotes $\int_I h(\alpha) d\alpha$ for any integrable h]; and $u: I \times \mathbb{R}_+^m \rightarrow \mathbb{R}$ is measurable such that $u(\alpha, x)$ is bounded in α for each x , and increasing in x for each α . The interpretation is that, for any α in I , $e_j(\alpha)$ is the initial endowment of commodity j of agent α , and $u(\alpha, x)$ is α 's utility for the bundle x .

An *allocation* is an integrable function $\chi: I \rightarrow \mathbb{R}_+^m$ satisfying $\int \chi = \int e$. Given a (*price*) vector p in the positive orthant \mathbb{R}_+^m , define α 's *budget set* $B^\alpha(p)$ by

$$B^\alpha(p) = \{x \in \mathbb{R}_+^m : p \cdot x \leq p \cdot e(\alpha)\}.$$

A *competitive equilibrium* (CE) is a pair $\langle p, \chi \rangle$ where $p \in \mathbb{R}_+^m$ is a price vector and χ is an allocation such that $\chi(\alpha)$ is optimal in α 's budget set for (almost) all α , i.e.,

$$\chi(\alpha) \in B^\alpha(p),$$

and

$$u(\alpha, \chi(\alpha)) = \max \{u(\alpha, x) : x \in B^\alpha(p)\}.$$

An allocation χ is called *competitive* if there is a price vector p such that $\langle p, \chi \rangle$ is a CE.

Our main theorem asserts that certain allocations are ‘approximately’ competitive. Here is the precise notion:

Definition 1. An allocation χ is called ε -competitive if there is a price vector p such that

- (a) $u(\alpha, \chi(\alpha)) \geq \max \{u(\alpha, (1-\varepsilon)x) : x \in B^\alpha(p)\}$,
- (b) $\mu(S_\varepsilon) \leq \varepsilon$, where $S_\varepsilon \equiv \{\alpha : p \cdot \chi(\alpha) \geq p \cdot (e(\alpha) + \varepsilon)\}$.

Condition (b) says that the allocations are approximately in the budget sets, and condition (a) says that they are approximately optimal.

In a sense to be made precise later, (b) is ‘asymptotically’ implied by (a). We refer the reader to the end of section 5 for a discussion of this notion.

3. The market mechanism

For each agent α there is a set M_α of ‘moves’ to be considered potentially available to α .

Let \mathcal{M} be a set of maps f from I to $\bigcup_{\alpha \in I} M_\alpha$ such that $f(\alpha) \in M_\alpha$ for each α (where \bigcup denotes disjoint union). For any such f and $a \in M_\alpha$, denote by $f|_\alpha a$ the same map as f except that $f(\alpha)$ is replaced by a . We assume that \mathcal{M} has the property $f|_\alpha a \in \mathcal{M}$ for any $f \in \mathcal{M}$, $\alpha \in I$, $a \in M_\alpha$. This will permit us to consider unilateral deviations by a single trader. In many cases all the M_α are a fixed measurable space, and \mathcal{M} is the set of all measurable maps from I to this space. Here the property above is immediate.

A *move-selection* f results in (a) a formation of *prices* $p(f)$ in $P = \bigcup_{K \subset I, m} \mathbb{R}_{++}^K$; and (b) an assignment of *trades* $t(f, \alpha) \in \mathbb{R}^m$ to the agents α such that $\int t(f, \alpha) d\alpha = 0$. A positive (resp. negative) component of $t(f, \alpha)$ represents a purchase (resp. sale) of the corresponding commodity by α .

As discussed in the introduction, it may be the case [e.g. Amir et al. (1990), Dubey and Shapley (1976)] that, for certain choices of f , not all prices are formed. Thus P represents the set of all possible partial price vectors, and \mathbb{R}_{++}^0 is understood to be the ‘wholly undefined vector’. Let

$$\tilde{P}(f) = \{p \in \mathbb{R}_{++}^m : p_i = p_i(f) \text{ for every } i \in I_m \text{ for which } p_i(f) \text{ is defined}\}.$$

Thus $\tilde{P}(f) \subset P$ is the set of complete prices that are compatible with f . [And $p(f) \in \tilde{P}(f) \Leftrightarrow p(f) \in \mathbb{R}_{++}^m$.]

If f is played, and $p(f) \notin \tilde{P}(f)$, we can complete $p(f)$ by announcing

'virtual' prices, and obtain $p \in \tilde{P}(f)$. What is an agent to conjecture about trades when faced with f and p ? This is specified by an extension of the domain of the mechanism. We assume that, for each α in I , there is a function

$$\phi^\alpha: \mathcal{M} \times M_\alpha \times P \rightarrow \mathbb{R}^m$$

such that his conjectures are consistent with the actual performance of the mechanism, i.e.,

$$t(f, \alpha) = \phi^\alpha(f, f(\alpha), p(f)) \quad \text{if } p(f) \in \tilde{P}(f).$$

[Note: The extension ϕ^α becomes irrelevant if, to begin with, $p(f) \in \tilde{P}(f)$ for all $f \in \mathcal{M}$].

Assumption 1 (Budget Balance). For each α in I , f in \mathcal{M} , a in M_α , and p in $\tilde{P}(f)$, we have $\phi^\alpha(f, a, p) \cdot p = 0$.

Notice that if $p(f) \in \tilde{P}(f)$, then $\int \phi^\alpha(f, f(\alpha), p(f)) \cdot p(f) \, d\alpha = 0$, since $\int t(f, \alpha) \, d\alpha = 0$. In other words, budget balance is perforce true *on the aggregate* at any play f (at which all prices form). One might interpret the above assumption to say that the mechanism does not assign favourable trades [i.e. ones with $\phi_\alpha(f, a, p) \cdot p > 0$] to some agents at the expense of others. This is a property that holds across all the models we have referenced.

Before proceeding, we need to discuss the feasibility of move selections. For any α in I the set M_α represents the *potential* set of moves available to α . Once every α sends $f(\alpha) \in M_\alpha$ to the mechanism, it produces trades $t(f, \alpha)$, with $\int t(f, \alpha) \, d\alpha = 0$ and in keeping with the conditions embodied in Axiom I. These represents, so to speak, the *public* (observable) constraints on the mechanism. But it may well happen that the trade $t(f, \alpha)$ assigned to agent α is one that α cannot honour. Indeed the feasibility of $t(f, \alpha)$ will depend on the bundle of commodities that α happens to have on hand at the time (e.g. he cannot sell more than he has); and, further, may depend on prices as well as other macroaggregates determined by f (e.g. if α had to spend money for purchases, the total value of his purchases evaluated at the market prices cannot exceed cash on hand). These constraints are *private* (unobservable) and the mechanism is blind to them. It assigns trades no matter which move selection is sent to it. [See Dubey et al. (1980) for a further discussion of this point]. But since we are interested in trades that can be carried out, it is necessary to distinguish the *subsets* of M_α that are *feasible* for α in different circumstances. To this end, it is convenient to introduce the map

$$\Psi: I \times \mathbb{R}_+^m \times P \times \mathcal{M} \rightarrow \{\text{nonempty subsets of } \mathcal{M}\},$$

where $\Psi(\alpha, x, p, f) \subset M_\alpha$ for each α , and is interpreted as the *set of moves feasible for α* , when he has the bundle x on hand, prices p prevail and *others* play in accordance with f . We naturally assume

$$a \in \Psi(\alpha, x, p, f) \Rightarrow x + t(\alpha, f) \in \mathbb{R}_+^m,$$

which is only to say that, to be feasible, α cannot offer to sell more than he has. (But the point of introducing Ψ in this generality is to allow for more, far severer, constraints on the trades that agents can conduct in the mechanism.)

While Assumption 1 holds equally for models with finitely many agents [see Amir et al. (1990), Benassy (1986), Dubey (1982), Dubey and Shubik (1978, 1980), Peck and Shell (1985), Postlewaite and Schmeidler (1978) and Shapley and Shubik (1977)], the next one is special to the continuum models.

Assumption 2. (Non-Atomicity). If $f = g$ almost everywhere, then

- (a) $f \in \mathcal{M} \Rightarrow g \in \mathcal{M}$, and $p(f) = p(g)$,
- (b) $f(\alpha) = g(\alpha) \Rightarrow t(f, \alpha) = t(g, \alpha)$,
- (c) $\Psi(\alpha, x, p(f), f) = \Psi(\alpha, x, p(g), g)$ for all α , provided $p(f) \in \tilde{P}(f)$.

The next assumption simply says that it is always possible for an agent to obtain the 0-trade vector.

Assumption 3 (No-Trade Option). For any $\{\alpha, x, p, f\}$, there is a in $\Psi(\alpha, x, p, f)$ such that $t(f|_a, \alpha) = 0$.

The final assumption is the most significant. It is best stated in terms of the following notions.

Definition 2. Commodity i is linked to j ($i \rightarrow j$) in the mechanism, if there exists $\eta > 0$ such that for all $\bar{\eta} < \eta$, $p \in \mathbb{R}_+^{m++}$, x in \mathbb{R}_+^l with $x_i > \bar{\eta}$, α in I and f in \mathcal{M} : there is an a in $\Psi(\alpha, x, p, f)$ for which

$$\phi_k^z(f, a, p) = \begin{cases} 0, & k \neq i, j \\ -\bar{\eta}, & \text{if } k = i \end{cases}$$

Simply put, $i \rightarrow j$ means that it is always possible for an agent, who has i , to give up a small amount (namely $\bar{\eta}$) of it in exchange for *only* j . [More precisely, we require this trading opportunity to be available in the mechanism when $p(f) \in \tilde{P}(f)$; otherwise, for any completion $p \in \tilde{P}(f)$ of $p(f)$, an agent should be able to *conjecture* this opportunity in ϕ^a and $\Psi(\alpha, \dots)$.]

Definition 3. Commodity i is connected to j (in the mechanism), if there is a chain $i = i_0 \rightarrow \dots \rightarrow i_k = j$.

Assumption 4 (Connectivity). For each pair (i, j) of distinct commodities in $I_m \times I_m$, i is connected to j .

4. The generalized game

The game is played for T periods as follows. In period 1, all agents select moves f_1 such that, for all α ,

$$f_1(\alpha) \in \Psi(\alpha, e(\alpha), p(f_1), f_1).$$

In period $\tau + 1$, the agents observe f_1, \dots, f_τ (modulo null sets) and their own past actions and choose $f_{\tau+1}$ such that, for all α ,

$$f_{\tau+1}(\alpha) \in \Psi(\alpha, e(\alpha) + \sum_{\sigma=1}^{\tau} t(f_\sigma, \alpha), p(f_{\tau+1}), f_{\tau+1}).$$

Note that $\Psi(\dots)$ is the set of moves feasible for α . His strategy is to select a move at each node that he can distinguish. In keeping with the non-atomicity assumption, we require that he observes others' moves only modolu null sets.

Let g_1, \dots, g_T (where each $g_\sigma \in \mathcal{M}$) be the moves made along the play induced by the strategies f_1, \dots, f_T . Then the payoff to α of this play is

$$u\left(\alpha, e(\alpha) + \sum_{\sigma=1}^T t(g_\sigma, \alpha)\right).$$

A strategic equilibrium (SE) is a choice of strategies such that no agent can improve his payoff by unilateral (feasible) deviations.

It may happen that, for several σ , the moves g_σ fail to produce complete prices, i.e. $p(g_\sigma) \notin \tilde{P}(g_\sigma)$. In this event was can 'open up' trading opportunities by completing $p(g_\sigma)$ to $p^\sigma \in \tilde{P}(g_\sigma)$ for all such σ . But then each agent is free to conjecture that he can conduct all the trades implies by $\{g_\sigma, p^\sigma\}_{\sigma=1}^T$. If, nevertheless, (almost) all traders α choose to stick to $g_\sigma(\alpha)$ for $1 \leq \sigma \leq T$, we will say that the S.E. is open. Formally, given an S.E. with induced play (g_1, \dots, g_T) , we say that the S.E. is open if there exist p^1, \dots, p^T in \mathbb{R}_{++}^m such that:

- (i) $p^\sigma \in \tilde{P}(g_\sigma)$ for $\sigma = 1, \dots, T$.
- (ii) for almost all α in I , α 's strategy is a best response even when α conjectures that he could trade in period σ at the prices p^σ , i.e., when he

conjectures that any choice $a \in \Psi(\alpha, x, p^\sigma, g_\sigma)$ is feasible (when he has x on hand) and yields the trade $\phi^\sigma(g_\sigma, a, p)$.

The idea is that when the mechanism fails to produce prices for some commodities, we can nevertheless announce virtual prices which will sustain the same actions by the agents.

Here is the main theorem. (Recall the notion of ε -competitive from Definition 1 of section 3.)

Theorem 1. Suppose Assumptions 1–4 hold. Further suppose for each pair of distinct commodities (i, j) , there is a non-null set of agents such that for all α in this set we have

$$e_i(\alpha) > 0, \tag{1}$$

$$\partial u^\alpha / \partial x_j \text{ is bounded away from } 0 \text{ in } \mathbb{R}_+^m. \tag{2}$$

Then for each $\varepsilon > 0$ there is a T^* such that, if $T \geq T^*$, any open SE of the T -period generalized game gives ε -competitive allocations.

[The conditions (1) and (2) say that for any pair of commodities (i, j) there is a nonnegligible set of agents who have i and want j .]

For the proof we need a couple of Lemmas.

Lemma 1. Let S be a (measurable) subset of I with Lebesgue measure $\mu(S) > 0$, and let A be a fixed vector in \mathbb{R}_+^m . Then there is a constant $C > 0$, such that if χ is any function from S to \mathbb{R}_+^m with $\int_S \chi \leq A$, then

$$2\mu(\{\alpha \in S : u(\alpha, \chi(\alpha)) > C\}) < \mu(S).$$

Proof. Let $C = \sup \{u(\alpha, 2mA/\mu(S)) \mid \alpha \in S\}$, and let S' be the set $\{\alpha \in S \mid u(\alpha, \chi(\alpha)) > C\}$. Then $\alpha \in S'$ implies $\chi_i(\alpha) > 2mA_i/\mu(S)$ for some i . Thus there is at least one fixed commodity i and a subset S'' of S' such that $\mu(S'') \geq \mu(S')/m$ and $\chi_i(\alpha) > (2mA_i)/\mu(S)$ for all α in S'' .

Now $A_i \geq \int_{S''} \chi_i > \mu(S'') 2mA_i/\mu(S) \geq 2\mu(S'') A_i/\mu(S)$ which yields the desired result. \square

Lemma 2. There is an integer T_0 and a positive real number R (both independent of T) such that if (p^1, \dots, p^T) are the prices generated at any open SE of the T -period game, we have, for each pair (i, j) ,

$$|\{\tau \mid p_i^\tau/p_j^\tau > R\}| < T_0. \tag{3}$$

(In other words, for all but a finite number (T_0) of time periods, all price ratios lie in the fixed compact set $[1/R, R]$.)

Proof. It suffices to establish the lemma for a fixed, linked pair of commodities (i, j) . (The general case then follows by the connectivity Assumption 4.)

By the assumption (1) in Theorem 1 we can find a positive number δ and a (non-null) set S such that $e_i(\alpha) > \delta$ for all α in S . Let $A = \int_I e$ and let C be the constant described in Lemma 1.

Now by assumption (2) of the theorem, there is a number E such that for any vector $z \geq 0$ with $z_j > E$, we have $u(\alpha, z) \geq C$ for all α in S .

Let $R = E/\delta$ and choose $0 < \bar{\eta} < \min\{\delta, \eta\}$ where η is as in Definition 2. Let T_0 be an integer greater than $\delta/\bar{\eta}$; and suppose, by way of contradiction, that (3) fails to hold. Then by the openness of the SE, and by Assumptions 1, 2, 3, 4, each α in S can convert δ units of commodity i into at least $R\delta = E$ units of commodity j . Consequently, if χ is the SE allocation, we have $u(\alpha, \chi(\alpha)) \geq C$ for all α in S . But since $\int_S \chi \leq \int_I \chi = \int_I e = A$, this is not possible in view of Lemma 1; and so (3) must hold. \square

We are now ready to prove Theorem 1. The main point is that Lemma 2 implies that as $T \rightarrow \infty$, the price ratios in the T -period game will have a non-zero limit point, which will be a candidate for a competitive price. The actual proof is a more careful, finite version of this idea.

Fix R as in Lemma 2, and define the sets

$$\begin{aligned}
 \Delta &= \{p \in \mathbb{R}_+^m \mid \sum_i p_i = 1 \text{ and } p_i/p_j \leq R \text{ for all pairs } (i, j)\}, \\
 Q &= \{y \in \mathbb{R}_+^{m^2} \mid 1/R \leq y_{ij} \leq R \text{ for } (i, j) \in I_m \times I_m\}.
 \end{aligned}
 \tag{4}$$

Also, for each commodity pair (i, j) , let us fix a chain (as in Definition 3) such that $i = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_k = j$, and define the functions on Q given by

$$F(y; i, j) = y_{i_0 i_1} y_{i_1 i_2} \dots y_{i_{k-1} i_k} \text{ and } F(y) = \max_{i, j} F(y; i, j).
 \tag{5}$$

Note that F is continuous, and hence uniformly continuous on the (compact) set Q .

Proof of Theorem 1. Given $\varepsilon > 0$ (as in the Theorem), we can find a $\zeta > 0$, such that for y, y' in Q , $|y - y'| < \zeta$ implies $|F(y) - F(y')| < \varepsilon$, where $|\cdot|$ denotes the l_1 -norm.

Recall the budget set $B^\alpha(p)$ defined in section 2 and let $B^\alpha = \bigcup_{p \in \Delta} B^\alpha(p)$; and $B = \bigcup_{\alpha \in I} B^\alpha$. Then by (4) and the boundedness of $\{e(\alpha) \mid \alpha \in I\}$ it follows that B is bounded. Let

$$X = \{\max x_j \mid 1 \leq j \leq m \text{ and } x \in B\}.
 \tag{6}$$

Let N_1 be the smallest integer greater than R/ζ and divide Q into N_1^m equal hypercubes each of which has edges of length $(R-1/R)/N_1 < \zeta$. We will show that if T is any integer greater than $T_0 + mN_1^m(X/\bar{\eta})^m = T^*$ then open SE-allocations in the T -period game are ε -competitive.

To see this, we observe that Lemma 1 and the choice of N_1 imply that for some $N \geq m(X/\bar{\eta})^m$, there are N time periods in which the price ratios lie inside a fixed hypercube of edge ζ in Q . Let these periods be $\{\tau_1, \dots, \tau_N\}$ and to simplify notation, let us write p for p^{τ_1} .

Fix a player α and let $x \in B^x(p)$. Then to finish the proof of the theorem, it suffices to show that α can achieve the bundle $(1-\varepsilon)x$ (or more) by an appropriate strategy. We describe such a strategy below.

Let $t = e(\alpha) - x$, and suppose without loss of generality that $t_1, \dots, t_k \geq 0$ and $t_{k+1}, \dots, t_m \leq 0$; i.e. α sells $1, \dots, k$ and buys $k+1, \dots, m$, in the game.

Suppose α deviates to the following strategy: He buys and sells only in periods $\{\tau_1, \dots, \tau_N\}$. (This is possible by Assumption 3). He begins by converting t_1 units of good 1 to good m using the $(1, m)$ -chain. If (i, j) is a link in this chain, then (6) (in conjunction with Assumption 4) implies that the conversion of i to j may be effected in less than $(X/\bar{\eta})$ steps. Consequently, in at most $(X/\bar{\eta})^m$ steps he can either finish selling t_1 units of 1 or will have bought $\lfloor (1-\varepsilon)t_m \rfloor$ units of m . In the first case he starts to use the $(2, m)$ chain, and in the second case the $(1, m-1)$ chain, etc. This procedure may be finished in $N(\geq m(X/\bar{\eta})^m)$ steps.

By the choice of ζ and (5) we see (using Assumption 1) that each time α converts s units of good i to good j in the (i, j) -chain he gets at least $(1-\varepsilon)sp_j/p_i$ units of good j .

Since repeated trading in the manner described above at the fixed price vector p would have got him exactly to x , we see from the above that he will manage to buy at least $(1-\varepsilon)\lfloor t_{k+1} \rfloor, \dots, (1-\varepsilon)\lfloor t_m \rfloor$ of goods $k+1, \dots, m$ by selling at most $\lfloor t_1 \rfloor, \dots, \lfloor t_k \rfloor$ units of goods $1, \dots, k$. This finishes the proof of part (a) of Definition 1. As remarked there, part (b) follows. See the end of section 4 for details. \square

The converse is easier, but requires an additional assumption on the mechanism:

Assumption 5 (Nondegeneracy). Given prices $p \in \mathbb{R}_+^m$ and trades $\tau: I \rightarrow \mathbb{R}^m$ such that $\int \tau = 0$, $e(\alpha) + \tau(\alpha) \in \mathbb{R}_+^m$ for $\alpha \in I$ and $p \cdot \tau(\alpha) = 0$; there exist feasible moves f_1, \dots, f_T which produces prices p in each round [i.e. $p(f_\sigma) = p$ for $1 \leq \sigma \leq T$] and yield τ as the cumulative trade (i.e. $\sum_{\sigma=1}^T t(f_\sigma, \alpha) = \tau(\alpha)$ for $\alpha \in I$).

Theorem 2. Any competitive allocation can be achieved as an open SE in finitely many rounds of trade.

Proof. This is obvious. Clearly if p is a competitive price vector, then $p > 0$. Chose R such that $p_i/p_j < R$ for all (i, j) .

Construct $X = X(R)$ as in the proof of Theorem 1. Announce the prices p repeatedly in each of the $m(X/\eta)^m$ periods, and scale the competitive trades to make the sales no more than η in each period. Then define strategies in the game tree to be consistent with the above trades (i.e., they can be arbitrary off the play that gives these trades). Since the player-set is nonatomic, these strategies constitute an SE. \square

[Note: If the mechanism has the property that prices never fail to form, i.e., $p(f) \in \mathbb{R}^m_{++}$ for all $f \in \mathcal{M}$, then every S.E. is obviously open. We reiterate that the whole discussion of conjectures ϕ^α is moot in this case].

5. Concluding comments

5.1. Examples

Example 1 (Cournot). We consider a class of market mechanisms with quantities as moves. The central notion is that of a trading-post for a pair of distinct commodities (i, j) . Suppose the agents send $0 \leq q_i(\alpha)$ and $0 \leq q_j(\alpha)$ to the post. Write $p_{ij} = \int q_i(\alpha) d\alpha / \int q_j(\alpha) d\alpha$ if both of these integrals exist and are positive. In this case the agent α gets back $q_j(\alpha)p_{ij}$ and $q_i(\alpha)/p_{ij}$ of commodities i and j respectively from the post. If either one of the integrals is 0, then prices do not form and we assume that the agents get back what they sent to the post.

Let there be $m+1$ commodities in the economy. Consider a graph on $m+1$ nodes with arc (i, j) if the trading post (i, j) exists. Then the connectivity Assumption 4 is simply the requirement that the graph be connected!

If the graph is a *spanning tree* we can choose one of the commodities (say $m+1$) as numeraire and determine prices $p_1, \dots, p_m, p_{m+1} = 1$ in terms of $m+1$. It is easily checked that in *this* case the mechanism satisfies all our assumptions. (If the graph is *not* a tree, then prices may be ‘inconsistent’ i.e., $p_{ij}p_{jk} \neq p_{ik}$, [see Amir et al. (1990)] and our theorems are not directly applicable. However, they are still true, and may be proved by similar arguments.)

A natural case occurs when one of the commodities plays the role of ‘money’, i.e. the stipulated means of exchange. Then the graph looks like a ‘rooted’ tree with an arc connecting each of the nodes $1, \dots, m$ to $m+1$.

This is the Shapley–Shubik mechanism of Shapley and Shubik (1977). Here we can interpret an increase in T to be tantamount to an increase in the ‘velocity’ of money; and Theorem 1 shows that this increase is enough to overcome the initial ‘illiquidity’.

It may be clarifying to specialize our notions of section 2 to this case. Thus

$M_\alpha = \mathbb{R}_+^{2m}$ for all $\alpha \in I$, and we denote a typical element of M_α by $(b(\alpha), q(\alpha)) \in \mathbb{R}_+^m \times \mathbb{R}_+^m$. [Think of $b_i(\alpha)$ as the money sent by α to purchase i , $q_i(\alpha)$ as the quantity of i sent by α for sale].

Then $\mathcal{M} = \{(b, q) \mid b: I \rightarrow \mathbb{R}_+^m, q: I \rightarrow \mathbb{R}_+^m \text{ are integrable}\}$.

For $f \in \mathcal{M}$, $\alpha \in I$, $1 \leq i \leq m$, we have $p_i(f) = \int b_i / \int q_i$, and $t_i(f, \alpha) = b_i(\alpha) / p_i(f) - q_i$ if both integrals are positive, otherwise p_i is undefined, and $t_i = 0$. Also $p_{m+1}(f) = 1$ and $t_{m+1}(f, \alpha) = -\sum b_i(\alpha) + \sum p_i(f) q_i(\alpha)$, where the summation runs over the commodities for which $p_i(f)$ is defined.

For $x \in \mathbb{R}_+^{m+1}$, we have $\Psi(\alpha, x, f) = \{(b(\alpha), q(\alpha)) \in M_\alpha \mid \sum_{i=1}^m b_i(\alpha) \leq x_{m+1}, q_i(\alpha) \leq x_i\}$.

Finally, for $p \in \mathbb{R}_+^{m+1}$, we have $\phi_i^\alpha(f, (b(\alpha), q(\alpha)), p) = b_i(\alpha) / p_i - q_i$; and $\phi_{m+1}^\alpha(f, (b(\alpha), q(\alpha)), p) = -\sum_{i=1}^m b_i(\alpha) + \sum_{i=1}^m p_i q_i(\alpha)$.

Example 2 (Bertrand). We present the ‘continuum’ version of one of the price–quantity strategic market games of Dubey (1982). There are $m+1$ commodities and m trading posts as before. Agents make announcements of the form ‘if the price of i is p_i or more, I am prepared to sell up to q_i ; and if the price is \tilde{p}_i or less, I am prepared to buy up to \tilde{q}_i units’. Thus $M_\alpha = \mathbb{R}_+^{4m}$, and M consists of measurable maps $f = (p, q, \tilde{p}, \tilde{q}): I \rightarrow \mathbb{R}_+^{4m}$. Given a ‘price’ $\gamma \geq 0$, the sellers and buyers of commodity i are the sets

$$I_i(r, f) \equiv \{\alpha \in I \mid p_i(\alpha) \leq r\} \quad \text{and} \quad \tilde{I}_i(r, f) \equiv \{\alpha \in I \mid \tilde{p}_i(\alpha) \geq r\},$$

respectively; and the potential supply and demand are given by

$$S_i(\gamma, f) \equiv \int_{I_i(\gamma, f)} q_i(\alpha) \quad \text{and} \quad D_i(\gamma, f) \equiv \int_{\tilde{I}_i(\gamma, f)} \tilde{q}_i(\alpha),$$

respectively. These are increasing and decreasing functions of γ . A price for i forms if $D_i(\gamma, f) \geq S_i(\gamma, f)$ for some $\gamma > 0$. In this case let $p_i(f) = \inf\{\gamma \mid D_i(\gamma, f) \geq S_i(\gamma, f)\}$. If further $D_i(p_i(f), f) = S_i(p_i(f), f)$, then agents α in $I_i(p_i(f), f)$ and $\tilde{I}_i(p_i(f), f)$ sell and buy their announced amounts at the price $p_i(f)$. If $D_i(p_i(f), f) > S_i(p_i(f), f)$, a rationing mechanism must be employed, the exact nature of which is not too relevant to the present discussion (e.g. simply scale down the demands). Payments must be made in the $(m+1)$ th commodity, money. [Thus α 's strategy must satisfy $q_i(\alpha) \leq e_i(\alpha)$ for $1 \leq i \leq m$; and $\sum_{i=1}^m \tilde{p}_i \tilde{q}_i(\alpha) \leq e_{m+1}(\alpha)$.] To complete the description, we further stipulate that if prices do not exist in a market, no trade takes place.

An important feature of this mechanism is the *inherent* discontinuity of outcomes with respect to strategies. If a seller α raises his price $p_i(\alpha)$ above $p_i(f)$, he'll suddenly find himself ‘out’ of the market; similarly for buyers who lower prices. Thus this mechanism does not quite fit into the framework of Dubey et al. (1980).

We leave it to the reader to check that it does, however, fit into the current framework.

5.2. Approximate CEs

We first remark that an allocation χ is competitive if and only if there is a price vector p such that for almost all α :

$$u(\alpha, \chi(\alpha)) \geq \max \{u(\alpha, z) \mid z \in B^2(p)\}.$$

The condition is clearly necessary. On the other hand, if it holds then we must have $p \cdot \chi(\alpha) \geq p \cdot e(\alpha)$ for each α . Since $\int \chi = \int e$, equality must hold for (almost) all α ; which shows that (χ, p) is a CE.

To complete the proof of Theorem 1, we give a similar argument for ε -competitive allocations, deducing (b) of Definition 1 from part (a).

Let R be as in Lemma 2, write C for $\int (e_1(\alpha) + \dots + e_m(\alpha))$, and define $\varepsilon_1 = \min \{\varepsilon, \varepsilon^2/R^2C\}$. As in the proof of Theorem 1, choose T^* large enough to ensure that all SE's satisfy $p \cdot \chi(\alpha) \geq (1 - \varepsilon_1)p \cdot e(\alpha)$. Let S_ε be as in Definition 1(b) and let S'_ε be its complement. Then we have

$$\int p \cdot e = \int p \cdot \chi = \int_{S_\varepsilon} p \cdot \chi + \int_{S'_\varepsilon} p \cdot \chi \geq \int_{S_\varepsilon} p \cdot (e + \varepsilon) + (1 - \varepsilon_1) \int_{S'_\varepsilon} p \cdot e;$$

which implies that

$$\int_{S_\varepsilon} p \cdot \varepsilon \leq \varepsilon_1 \int_{S'_\varepsilon} p \cdot e \leq (\varepsilon^2/R^2C) \int p \cdot e.$$

Since $p \in Q$ [see eq. (4)], we get $\mu(S_\varepsilon)(\varepsilon/R) \leq RC(\varepsilon^2/R^2C)$ which simplifies to yield $\mu(S_\varepsilon) \leq \varepsilon$ as desired.

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