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Explicit Hilbert spaces for certain unipotent representations

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Summary. Let G be the universal cover of the group of automorphisms of a symmetric tube domain and let $P = LN$ be its Shilov boundary parabolic subgroup. This paper attaches an irreducible unitary representation of G to each of the (finitely many) L-orbits on $\mathfrak{n}^*$.

The Hilbert space of the representation consists of functions on the orbit which are square-integrable with respect to a certain L-equivariant measure. The representation remains irreducible when restricted to P, and descends to a quotient of G which is, at worst, the *double* cover of a linear group.

If the L-orbit is *not* open (in $\mathfrak{n}^*$), the construction gives a unipotent representation of G.

Introduction

Let G be the universal covering group of the group of automorphisms of a symmetric tube domain and let $P = LN$ be its Shilov boundary parabolic subgroup. Then L has finitely many (coadjoint) orbits on $\mathfrak{n}^*$, and each orbit $\mathcal{O}$ has an L-equivariant measure $d\mu$.

The main result of this paper is the construction of an irreducible unitary representation of G on $L^2(\mathcal{O}, d\mu)$; or rather on the Hilbert space H consisting of those tempered distributions on $\mathfrak{n}$ which are Fourier transforms of $\psi d\mu$ for some ψ in $L^2(\mathcal{O}, d\mu)$.

More precisely, we describe a *non-unitary* character of the opposite parabolic $\bar{P}$ and a $(\mathfrak{g}, K)$-submodule V of the induced representation such that, if we regard V as a subspace of $C^\infty(\mathfrak{n})$ (via the Gelfand–Naimark decomposition and the exp map), then we have

Theorem. *V is a dense subspace of H, and the restriction of the norm is $(\mathfrak{g}, K)$-invariant.*

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We now describe the setting, and our results, in greater detail.

First of all, the L-orbits on $\mathfrak{n}^*$ are parametrized by pairs of non-negative integers (p, q) whose sum is at most n. If one of p or q is 0, we will say that the corresponding orbit is *semi-definite*, otherwise we will call it an *indefinite* orbit.

For the semi-definite orbits, the representations are highest weight modules whose *unitarizability* was first proved in [W]. It was shown in [RV], by an explicit calculation of the associated reproducing kernel, that these representations admit realizations on suitable Hilbert spaces of holomorphic functions on the tube domain.

For the indefinite orbits the unitarizability of the corresponding representations was proved in [G], and one *expects* to be able to realize these representations on appropriate cohomology spaces of certain complex varieties which are *not* Stein manifolds. But in this setting, it is unclear how the inner product should be defined – certainly, one will not have anything as nice as a reproducing kernel function!

The point of this paper is that while the cohomology spaces might themselves be somewhat obscure, one can profitably study them via their "boundary values", replacing the complex analysis by real analysis. This is similar in spirit to [SS].

The main problem then is to prove the invariance of the L^2-norm with respect to a suitable multiplier representation of G. For the semi-definite orbits this in [RV], while if G is classical, it may be deduced from the theory of the oscillator representation as in [L]. Various special cases have also been considered before in [JV, O, KV] and elsewhere. But none of these methods is sufficiently general for our purposes.

Our approach consists of exploiting the presence of a one-dimensional lowest K-type in V. We first calculate the Fourier transform of the corresponding function σ and show that it lies in H. The P-invariance of the norm in H is immediate and, since $G = PK$, the G-invariance follows from a certain "uniqueness result" for P-invariant norms on H, which in turn is a simple consequence of a recent result of Poguntke [P] on the existence of rank 1 projections in the L^1 group algebra.

The method of this paper transcends the exigencies of the situation, and subsequent papers will explore two further applications. The first will be to the construction of Hilbert spaces for small unipotent representations of a more general kind, and the second to a possible extension of the theory of dual pairs to certain exceptional groups.

1 Algebraic preliminaries

1.1 The Lie structure

For the most part we will follow the notation of [S]. Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible Hermitian symmetric pair of tube type, and let $\mathfrak{h}^s = \mathfrak{a}^s \oplus \mathfrak{t}^s$ be a maximally split Cartan subalgebra of $\mathfrak{g}$. It is known that the restricted root system is of type C_n where $n = \dim(\mathfrak{a}^s)$. Thus we may choose a basis $\varepsilon_1, \ldots, \varepsilon_n$ for $(\mathfrak{a}^s)^*$ such that $\Sigma(\mathfrak{g}, \mathfrak{a}^s) = \{\pm \varepsilon_i \pm \varepsilon_j\} \cup \{\pm 2\varepsilon_j\}$.

The root spaces for $\pm\varepsilon_i \pm \varepsilon_j$ have a common dimension which we denote by d. The root spaces for $\pm 2\varepsilon_j$ are one-dimensional. Thus to each $2\varepsilon_j$ we may attach in a standard manner, an S-triple $\{h_j, e_j, f_j\}$, contained in $\mathfrak{g}$. These S-triples commute, and $\{h = \sum h_j, e = \sum e_j, f = \sum f_j\}$ is also an S-triple.

The Cayley transform is the element $c = \exp\frac{\pi i}{4}(e+f)$ in $\mathrm{Ad}(\mathfrak{g}_c)$. Let $\mathfrak{t} = ic(\mathfrak{a}^s)$, then $\mathfrak{h} = \mathfrak{t} + \mathfrak{t}^s$ is a *compact* Cartan subalgebra for $\mathfrak{g}$ (and $\mathfrak{k}$); and $\{\gamma_i | \gamma_i = c\circ(2\varepsilon_i), i = 1, \ldots, n\}$ is a maximal set of (Harish-Chandra) strongly orthogonal roots in $\Sigma(\mathfrak{h}_c, \mathfrak{g}_c)$.

The eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}$ are -2, 0 and 2; let $\bar{\mathfrak{n}}$, $\mathfrak{l}$ and $\mathfrak{n}$ be the corresponding eigenspaces. Then $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are abelian and $\mathfrak{l} + \mathfrak{n}$ and $\mathfrak{l} + \bar{\mathfrak{n}}$ are maximal parabolic subalgebras.

Let G and K be the simply connected groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, and let $P = LN$ and $\bar{P} = L\bar{N}$ be the maximal parabolic subgroups of G corresponding to $\mathfrak{l} + \mathfrak{n}$ and $\mathfrak{l} + \bar{\mathfrak{n}}$. Then G/K is a symmetric tube domain of rank n and $G/\bar{P}$ is its Shilov boundary.

Now L (resp. K) has a unique positive (resp. unitary) character ν (resp. μ) whose differential is $\varepsilon_1 + \cdots + \varepsilon_n$ (resp. $\gamma_1 + \cdots + \gamma_n$). As remarked in [S], the restriction of μ to $L \cap K$ extends to a unique unitary character of L, also denoted by μ, which is trivial on the identity component of L. (This conflicts slightly with our use of "$d\mu$" for the equivariant measure, but will not lead to any confusion.)

1.2 The Jordan structure

Consider the adjoint action of $\mathfrak{l}$ on $\mathfrak{n}$. It is easy to see that the stabilizer of e is $\mathfrak{l}\cap\mathfrak{k}$. Consequently, the map $x \mapsto [x, e]$ is a $\mathfrak{l}\cap\mathfrak{k}$-isomorphism between $\mathfrak{l}\cap\mathfrak{p}$ and $\mathfrak{n}$. This enables one to define a commutative product on $\mathfrak{n}$ by setting $[x_1, e]\circ[x_2, e] = [[x_1, x_2], e]$.

It is shown in [K], that with respect to this product, $\mathfrak{n}$ becomes a simple, formally real, Jordan algebra with e as the unit, and with L and $L\cap K$ as its structure and automorphism groups, respectively. The action of the Cayley transform can be described in terms of the Jordan structure and we have $c\cdot x = (e+ix)(e-ix)^{-1}$ (see [KW] and [K]).

The (reduced) norm on $\mathfrak{n}$ (see [BK, Kap. II] is a polynomial φ, of degree n, which satisfies $\varphi(e) = 1$ and $\varphi(l\cdot x) = \nu^{-2}(l)\varphi(x)$ for all l in L.

In a similar manner, $\bar{\mathfrak{n}}$ becomes a Jordan algebra with f as its unit element. To avoid confusion, we will write x for a typical element of $\mathfrak{n}$, and y for an element in $\bar{\mathfrak{n}}$. Also we will write η for a typical function on $\mathfrak{n}$, and ψ for one on $\bar{\mathfrak{n}}$.

The Killing form on $\mathfrak{g}$ gives us a pairing between $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, which we will denote by $(,)$, *after normalizing it so that* $(e,f) = 1$. This pairing satisfies $(l\cdot x, l\cdot y) = (x, y)$, and allows us to consider $\bar{\mathfrak{n}}$ as the dual space of $\mathfrak{n}$.

1.3 Some results from [S]

We will write $(\pi_{\varepsilon,t}, I(\varepsilon,t))$ for the induced representation $\mathrm{Ind}_{\bar{P}}^{G}(\mu^{\varepsilon}\otimes\nu^{t})$ (normalized, C^∞-induction). By the Gelfand-Naimark decomposition and the exp map, $I(\varepsilon, t)$ can be realized on a subspace of $C^\infty(\mathfrak{n})$. The formulas for the action of P are particularly simple. N acts by translation, and since the modular function for $\bar{P}$ is ν^{-2r} where $r = 1 + \frac{d}{2}(n-1)$, we have for l in L

$$\pi_{\varepsilon,t}(l)\eta(x) = \mu^{\varepsilon}(l)\nu^{t-r}(l)\eta(l^{-1}\cdot x). \tag{1}$$

The K-types in $I(\varepsilon, t)$ have multiplicity one and their highest weights are of the form $\varepsilon\mu + \sum a_i\gamma_i$, where the a_i are integers with $a_1 \geqq \ldots \geqq a_n$.

For $p + q \leqq n$, put $\varepsilon = \frac{d}{4}(p-q)$, $t = 1 + \frac{d}{2}(n-1-p-q)$, and write (π_{pq}, I_{pq}) for $(\pi_{\varepsilon,t}, I(\varepsilon, t))$. Theorem 5 in [S] shows that I_{pq} has a unitarizable $(\mathfrak{g}, K)$-subquotient V_{pq}, whose K-types are

$$\left\{\frac{d}{4}(p-q)\mu + \sum_{i=1}^{n} c_i\gamma_i \,|\, c_1 \geqq \ldots \geqq c_n \in \mathbf{Z};\, c_{p+1} = \ldots = c_{n-q} = 0\right\}.$$

If $p + q < n$, these are the representations in Theorem 5C of [S], while for $p + q = n$ they are among those contained in the more general Theorem 5B. Moreover, from the description of the Jantzen filtration in §2 of [S], it follows that V_{pq} is a submodule of I_{pq}. Thus we may, and will, regard V_{pq} as a subspace of $C^{\infty}(\mathfrak{n})$.

2 Analytic results

2.1 Orbits and stabilizers

Satz XI.5.5. in [BK] shows that each element in $\bar{\mathfrak{n}}$ is $L \cap K$-conjugate to an element of the form $r_1 f_1 + \ldots + r_n f_n$, where the r_j are real numbers, unique up to permutation. Since $L = (L \cap K)A(L \cap K)$, it follows that the elements $f_{pq} = (f_1 + \ldots + f_p) - (f_{n-q+1} + \ldots + f_n)$ are a set of representatives for the L orbits on $\bar{\mathfrak{n}}$.

We describe next the stabilizer S_{pq} of f_{pq} in L, or, rather, its Lie algebra $\mathfrak{s}_{pq}$. For the semi-definite orbits, this is calculated in [RV] and the general case may be deduced by passing to the complexification. We omit the (easy) proof and content ourselves with stating the result.

If $p + q$ is *equal* to n, $\mathfrak{s}_{pq}$ is a real form of $\mathfrak{s}_{n0} = \mathfrak{l} \cap \mathfrak{k}$.

If $p + q$ is less than n, we proceed as follows: Write $\mathfrak{l}_{ij}$ for the $(\varepsilon_i - \varepsilon_j)$-root space in $\mathfrak{l}$. Let $\mathfrak{l}_1$ (resp. $\mathfrak{l}_2$) be the subalgebras of $\mathfrak{l}$ spanned by $\{h_i, \mathfrak{l}_{ij}\}$ such that both (resp. neither) of i and j belong to the set of indices $\{p+1, p+2, \ldots, n-q\}$; also let $\bar{\mathfrak{u}}$ be the subalgebra spanned by $\{\mathfrak{l}_{ij}\}$ such that i is in this set while j is *not*. Then $\mathfrak{l}_2 \cap \mathfrak{s}_{pq}$ is a real form of $\mathfrak{l}_2 \cap \mathfrak{k}$, and

$$\mathfrak{s}_{pq} = \mathfrak{l}_1 + (\mathfrak{l}_2 \cap \mathfrak{s}_{pq}) + \bar{\mathfrak{u}}. \tag{2}$$

We also need to consider $\mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}$ with $p + q \leqq n$, which is (clearly) contained in $\mathfrak{s}_{pq}$. In fact, $\mathfrak{l}_2 \cap \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}$ is compact, and

$$\mathfrak{s}_{p0} \cap \mathfrak{s}_{0q} = \mathfrak{l}_1 + (\mathfrak{l}_2 \cap \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}) + \bar{\mathfrak{u}}. \tag{3}$$

We now calculate the trace of the adjoint representations of $\mathfrak{s}_{pq}$ and $\mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}$. For semidefinite orbits, the trace vanishes on the last two factors of (2) and (3), since they are compact and nilpotent, respectively. For indefinite orbits, the second factor in (2) is no longer compact, but the trace still vanishes on it, since it is conjugate to $\mathfrak{l}_2 \cap \mathfrak{k}$ after *complexification.*

If $p + q = n$ the first factor is missing, and so $\mathfrak{s}_{pq}$ is unimodular. If $p + q < n$, we have to consider the trace of the adjoint action of $\mathfrak{l}_1$ on $\bar{\mathfrak{n}}$. Since the dimension of each $\mathfrak{l}_{ij}$ is d, we see that this trace is $d(p+q)(\varepsilon_p + \ldots + \varepsilon_{n-q})$. It follows that,

(4) $\quad \operatorname{trace}(\operatorname{ad} \mathfrak{s}_{pq}) = d(p+q)\nu|\mathfrak{s}_{pq}, \; \operatorname{trace}(\operatorname{ad} \mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}) = d(p+q)\nu|\mathfrak{s}_{p0} \cap \mathfrak{s}_{0q}.$

Let us write $\mathcal{O}_{pq}$ for the L-orbit of f_{pq}. The following Lemma is crucial:

Lemma. *If $p + q \leqq n$, then the S_{0q}-orbit of f_{p0} is open and dense in $\mathcal{O}_{p0}$.*

This is probably well known, but lacking a suitable reference, we shall sketch a proof. First of all, let $\varphi_1, \ldots, \varphi_n$ be the polynomials defined in Theorem 0 of [KS], for example. ($\varphi_n = \varphi$ and the other φ_j's are norm functions for smaller Jordan algebras.)

It is easy to see that the set of y in $\bar{\mathfrak{n}}$ for which $\varphi_1(y), \ldots, \varphi_p(y)$ are positive, and $\varphi_{p+1}(y) = \ldots = \varphi_n(y) = 0$, is open and dense in $\mathcal{O}_{p0}$. Finally, by a procedure analogous to Gaussian elimination, one may show that $\exp(\mathfrak{s}_{0q})$ acts transitively on this set.

2.2 *Measures and convolution*

In this section, we determine equivariant measures on the various orbits. The basic result for such measures is the following:

Suppose G is a Lie group and H is a closed subgroup whose modular function δ_H *extends to a character of* G. Then it is well known (and easy to prove) that G/H has a natural measure which transforms by the character δ_G/δ_H of G and satisfies, for $f \in C_c^\infty(G)$,

(5) $$\int_{G/H} \left(\int_H f(gh)dh \right) d_{G/H} = \int_G f(g) \frac{\delta_G(g)}{\delta_H(g)} dg.$$

Thus it follows from (4) that $\mathcal{O}_{pq} = L/S_{pq}$ has an L-equivariant measure $d\mu_{pq}$ which transforms by the character $\nu^{d(p+q)}$. For $p + q < n$, this is the measure given by (5); however for $p + q = n$, (5) gives an *invariant* measure, which we adjust by multiplying by a suitable power of the absolute value of the Jordan norm, φ.

It also follows from (4) that $S_{pq}/(S_{p0} \cap S_{0q})$ has an S_{pq}-invariant measure.

For the rest of this section, let us fix p and q with $p + q \leqq n$. To simplify the notation slightly, we will write $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$ for $\mathcal{O}_{pq}, \mathcal{O}_{p0}, \mathcal{O}_{0q}$ respectively, S, S_1, S_2 for their stabilizers, and $d\mu, d\mu_1, d\mu_2$ for their equivalent measures.

Now put $\mathcal{O}' = \mathcal{O}_1 \times \mathcal{O}_2$, $d\mu' = d\mu_1 \times d\mu_2$, and $S' = S_1 \cap S_2$, then $\mathcal{O}'$ is "almost" a single L-orbit. Indeed, the action of L on (f_{p0}, f_{0q}) gives an L-equivariant map from L/S' to $\mathcal{O}'$, and Lemma 2.1 implies that the complement of L/S' in $\mathcal{O}'$ is a set of μ'-measure zero.

If ψ is a function on $\mathcal{O}'$, we define $\mathscr{C}\psi$ to be the function on $\mathcal{O}$ given by:

$$\mathscr{C}\psi(lS) = \int_{S/S'} \psi(lsS')d_{S/S'}(sS')$$

where $d_{S/S'}$ is the S-invariant measure on S/S'.

Now if ψ_1 and ψ_2 are compactly supported smooth functions on $\mathcal{O}_1$ and $\mathcal{O}_2$ then we define their "convolution" by $\psi_1 * \psi_2 = \mathscr{C}(\psi_1 \times \psi_2)$. It easily checked that the (implicit) integral is convergent and yields a smooth function. Moreover (4) implies

$$\int_{\mathcal{O}} \psi_1 * \psi_2 \, d\mu = \left(\int_{\mathcal{O}_1} \psi_1 \, d\mu_1 \right) \left(\int_{\mathcal{O}_2} \psi_2 \, d\mu_2 \right). \tag{6}$$

2.3 The Fourier transform

Let (,) be the L-invariant pairing between $\bar{\mathfrak{n}}$ and $\mathfrak{n}$ described in §1.2. If dm is a tempered measure on $\bar{\mathfrak{n}}$, its Fourier transform is the tempered distribution on $\mathfrak{n}$ defined by $\int \exp(ix, y)\, dm(y)$.

For each orbit, we describe a special "Gaussian" function and calculate the Fourier transform of the associated measure. For the semi-definite orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, the Gaussians are defined to be $\mathbf{e}_1(y) = \exp(-e, y)$ and $\mathbf{e}_2(y) = \exp(e, y)$, respectively; while for the indefinite orbit $\mathcal{O}$, we define its Gaussian function by means of the convolution $\mathbf{e} = \mathbf{e}_1 * \mathbf{e}_2$.

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are smooth functions with rapid decay it follows easily that $\mathbf{e}$ is a smooth, $L^1 \cap L^\infty$-function on $\mathcal{O}$. In particular, $\mathbf{e}\,d\mu$ is a tempered distribution. We will now renormalize $d\mu_1$, $d\mu_2$, $d\mu$ so that $\int_{\mathcal{O}_1} \mathbf{e}_1 \, d\mu_1 = \int_{\mathcal{O}_2} \mathbf{e}_2 \, d\mu_2 = \int_{\mathcal{O}} \mathbf{e} \, d\mu = 1$.

Lemma. *The Fourier transform of* $\mathbf{e}\,d\mu$ *is* $\sigma(x) = \varphi(e - ix)^{-dp/2} \varphi(e + ix)^{-dq/2}$.

Proof. We first show that if $x = l \cdot e$, then

$$\int_{\mathcal{O}_1} \exp(-x, y)\, d\mu_1(y) = \varphi(x)^{-dp/2}. \tag{7}$$

To see this, observe that $(x, y) = (l \cdot e, y) = (e, l^{-1} \cdot y)$. Consequently, the left side of (7) becomes $\int_{\mathcal{O}_1} \exp(-e, l^{-1} \cdot y)\, d\mu_1(y) = \int_{\mathcal{O}_1} \mathbf{e}_1(y)\, d\mu_1(l \cdot y) = v(l)^{dp}$. On the other hand, $\varphi(x)^{-dp/2} = \varphi(l \cdot e)^{-dp/2} = v(l)^{dp}$, as well. This proves (7).

Now the L-orbit of e is open in $\mathfrak{n}$, and so by analytic continuation, (7) holds if we replace x by $x' - ix$ where $x' \in L \cdot e$, and $x \in \mathfrak{n}$ is arbitrary. Specializing to $x' = e$, we get $\varphi(e - ix)^{-dp/2} = \int_{\mathcal{O}_1} \exp(-e + ix, y)\, d\mu_1(y) = \int_{\mathcal{O}_1} \exp(ix, y)\mathbf{e}_1(y)\, d\mu_1(y)$. This proves the Lemma for $q = 0$ and the proof for $p = 0$ is analogous.

Finally, since $\exp(ix, y_1 + y_2) = \exp(ix, y_1)\exp(ix, y_2)$, (6) gives the identity

$$\int_{\mathcal{O}} \exp(ix, y)\mathbf{e}_1 * \mathbf{e}_2(y)\, d\mu = \left(\int_{\mathcal{O}_1} \exp(ix, y_1)\mathbf{e}_1(y_1)\, d\mu_1 \right) \left(\int_{\mathcal{O}_2} \exp(ix, y_2)\mathbf{e}_2(y_2)\, d\mu_2 \right),$$

from which the Lemma follows immediately. QED

3 The main result

3.1 Irreducibility on P_o

Let us fix p and q, and put $t = 1 + \frac{d}{2}(n - 1 - p - q)$, then (1) gives $\pi_{\varepsilon, t}(l)\eta(x) = \mu^{\varepsilon}(l) v^{-\frac{d}{2}(p+q)} \eta(l^{-1} \cdot x)$.

If η is the Fourier transform of a distribution of the form $\psi\, d\mu_{pq}$, then an easy calculation using the invariance of the pairing, and the equivariance of the measure $d\mu_{pq}$ shows that $\pi_{\varepsilon,t}(l)\eta$ is the Fourier transform of $\mu^{\varepsilon}(l)v^{-\frac{d}{2}(p+q)}\psi(l^{-1}\cdot y)\, d\mu_{pq}$.

If H is as in the introduction, it follows that $(\pi_{\varepsilon,t}, H)$ is a unitary representation of P.

Lemma. *$(\pi_{\varepsilon,t}, H)$ is irreducible even upon restriction to the identity component P_o.*

The argument is classical and we sketch the proof. Realize $\pi_{\varepsilon,t}$ on $L^2(\mathcal{O}, d\mu)$, via the Fourier transform and let T be a bounded intertwining operator. It suffices to show that T must be a constant.

The action of N consists of multiplication by characters on $\mathcal{O}$, and since these characters separate points, it follows from the Stone-Weierstrass theorem that T is itself the operator of multiplication by a bounded Borel function. Finally, since L_o acts transitively on $\mathcal{O}$, we see that this function is a constant, which proves the lemma.

3.2 The 1 dimensional K-type

Let us now fix p and q and consider π_{pq}, I_{pq}, and V_{pq} as in 1.3. For simplicity of notation we suppress the indices.

The next lemma provides the crucial connection between V and $\mathcal{O}$.

Lemma. *The function σ (of Lemma 2.3) lies in the K-type $\frac{d}{4}(p-q)\mu$ of V.*

Proof. Let $\sigma_{a,\varepsilon,t}$ be a function in the 1 dimensional K-type $(a+\varepsilon)\mu$ in $I(\varepsilon, t)$. From (1), we get $\pi_{0,r}(l)\eta(x) = \eta(l^{-1}\cdot x)$, and thus $\pi_{0,r}(l)\varphi = v^2(l)\varphi$. Since the Cayley transform maps $\mathfrak{l}_c$ to $\mathfrak{k}_c$ and $2v$ to μ, we see that $\sigma_{a,0,r} = \varphi^a(c\cdot x)$ and $\sigma_{a,\varepsilon,r} = \varphi^{a+\varepsilon}(c\cdot x)$.

Now, for arbitrary t, $\sigma_{a,\varepsilon,t}$ is obtained by multiplying $\sigma_{a,\varepsilon,r}$ by the spherical function in $I(0,t)$. This spherical function was calculated in Lemma 2.1.3 in [S1] and was shown to be $\varphi^{(t-r)/2}(e+x^2)$. Specializing to $t = 1 + \frac{d}{2}(n-1-p-q)$, and $a+\varepsilon = \frac{d}{4}(p-q)$ we get

$$\sigma = \varphi^{-\frac{d}{4}(p+q)}(e+x^2)\varphi^{\frac{d}{4}(p-q)}(c\cdot x). \tag{8}$$

Now $e + x^2 = (e+ix)(e-ix)$, and $c\cdot x = (e+ix)(e-ix)^{-1}$, and while in general the norm of the product of two elements in $\mathfrak{n}$ is *not* equal to the product of their norms, this *is* true for $(e+ix)$ and $(e-ix)$. Thus (8) simplifies to the formula of Lemma 2.3. QED

3.3 The proof of the main theorem

Let H be as in the introduction, then Lemma 3.2 implies that σ belongs to H. Let us write $(,)$ for the inner product on H, and $\langle , \rangle$ for the $(\mathfrak{g}, K)$-invariant form on V. Let $\mathscr{H}$ be the (abstract) Hilbert space closure of V with respect to $\langle , \rangle$.

Now let $\mathscr{A}(G)$ denote the convolution algebra of *smooth* L^1 functions on G, and consider the subspace $W = \pi(\mathscr{A}(G))\sigma$ of $\mathscr{H}$. It is easy to see that W consists of smooth functions on G, which are therefore determined by their restrictions to N. Moreover, since $G = P_o K$ and σ transforms by a character of K, it follows that $W = \pi(\mathscr{A}(P_o))\sigma$. Since (π, H) is a unitary representation, we see that $W \subseteq H$.

Thus W is a subspace of $H \cap \mathscr{H}$, and carries two $\pi(P_o)$-invariant forms. By the Lemma in the appendix, it follows that there is an $\mathscr{A}(P_o)$-invariant subspace W' of W, such that the two forms are proportional on W'. Since W' is dense in H, by considering its closure, we get an isometric, P_o-invariant, imbedding of H into $\mathscr{H}$. Finally, since W is G-invariant, and $\mathscr{H}$ is an irreducible representation of G, W is dense in $\mathscr{H}$; and since $W \subseteq H$, it follows that $H = \mathscr{H}$.

3.4 Concluding comments

1. The K-types of V_{pq} were described in 1.3. If $\frac{d}{4}(p-q)$ is an integer the representation actually descends to the adjoint group. If $\frac{d}{4}(p-q)$ is a half-integer, we still get a linear group, but if $\frac{d}{4}(p-q)$ is a quarter integer the representation is that of a metaplectic double cover. For $G = \mathrm{Sp}(n, \mathbf{R})$ we have $d = 1$, and H_{10} is the holomorphic part of the oscillator representation.

2. By combining Lemma 3.2 with Theorem 1 of [M], it may be shown that the associated variety of H_{pq} is the coadjoint G-orbit containing $\mathcal{O}_{pq}$. One may also check that if $p + q < n$, then H_{pq} is a unipotent representation in the sense of [V].

3. If $p + q = n$, then the orbit $\mathcal{O}_{pq}$ is open, and 2.2 shows that there is a one-parameter family of L-equivariant measures and Hilbert spaces. These spaces carry natural unitary representations of P and one may ask which of these extend to G. The answer, which is quite subtle, may be obtained by combining Theorems 5A and B of [S] with the techniques of this paper.

4. The group G is generated by P together with the element $\imath = \exp(e - f)$, which acts on the Jordan algebra by inversion, and generalizes the conformal inversion operator in Minkowski space. One can check that for η in $C^\infty(\mathfrak{n})$,

$$\pi_{pq}(\imath)\eta(x) = \chi_{pq}(x)\eta(-x^{-1}),$$

where χ_{pq} is a certain fractional power of the norm function. Our main result is equivalent to the unitarity of this operator on H_{pq}.

Appendix

Let P_o be a connected Lie group of type 1, with modular function δ, and let $\mathscr{A}$ denote the convolution algebra of smooth functions in $L^1(P_o)$. Then $\mathscr{A}$ is a $*$-algebra with respect to the operation $f^*(p) = \overline{f(p^{-1})}\delta(p^{-1})$.

Proposition. [P] *If (π, H) is an irreducible unitary representation of P_o, then there is a function f in $\mathscr{A}$, with $f = f^*$ such that $\pi(f)$ is an (orthogonal) projection of rank* 1.

Write $(,)$ for the inner product on H. Then, as observed by R. Howe, this proposition implies the following uniqueness result for $(,)$.

Lemma. *Let W be an non-zero, $\mathscr{A}$-invariant, (necessarily dense) subspace of H, and let $\langle,\rangle$ be any $\mathscr{A}$-invariant inner product on W. Then there is a non-zero, $\mathscr{A}$-invariant, (necessarily dense) subspace W' of W such that $\langle,\rangle$ is proportional to $(,)$ on W'.*

Proof. Choose f as in the proposition and fix $v \neq 0$ in the range of $\pi(f)$. Since W is dense in H, we can find a u in W such that $(u, v) \neq 0$. Now $f = f^*$ implies that $(\pi(f)u, v) = (u, \pi(f)v) = (u, v) \neq 0$. Consequently, $\pi(f)u \neq 0$, and so $\pi(f)u$ must be a *non-zero* multiple of v. It follows that v belongs to W.

Let $\alpha = \langle v, v\rangle/(v, v)$, we claim that $\langle w, v\rangle = \alpha(w, v)$ for *all* w in W. Indeed, the left side is $\langle w, \pi(f)v\rangle = \langle \pi(f^*)w, v\rangle = \langle \pi(f)w, v\rangle$. (The first equality follows from the invariance of $\langle,\rangle$, and the second holds since $f = f^*$.) Similarly, $(w, v) = (w, \pi(f)v) = (\pi(f)w, v)$. Since $\pi(f)w$ is a multiple of v, the claim follows.

Now if h is any function in $\mathscr{A}$, then $\langle w, \pi(h)v\rangle = \langle \pi(h^*)w, v\rangle = \alpha(\pi(h^*)w, v) = \alpha(w, \pi(h)v)$. The Lemma follows upon putting $W' = \pi(\mathscr{A})v$. QED

(It was pointed out by N. Wallach that if one considers subspaces that are merely P_o-invariant, then one *can* construct non-proportional P_o-invariant inner products.)

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