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Binomial Coefficients and Littlewood–Richardson Coefficients for Jack Polynomials

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In this paper, we consider translation and multiplication operators acting on the rings of symmetric and nonsymmetric polynomials and study their matrix coefficients with respect to the bases of Jack polynomials and interpolation polynomials. The main new insight is that the symmetric and nonsymmetric cases share a key combinatorial feature, that of a locally finite graded poset with a minimum element. This allows us to treat both cases in a simple and *unified* manner.

1 Introduction

In this paper, we consider translation and multiplication operators acting on the polynomial ring and study their matrix coefficients with respect to the bases of Jack polynomials and interpolation polynomials.

Let $\mathbb{F} = \mathbb{Q}(\alpha)$ be the field of rational functions in a parameter α. The nonsymmetric Jack polynomials [2, 8, 16] and interpolation polynomials [6, 19] are bases for the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$, respectively homogeneous and inhomogeneous, indexed by the set $\mathcal{C}_n$ of compositions of length $\leq n$. Their symmetric counterparts ([4, 12, 13, 21] and [7, 17]) are bases for the subring of symmetric polynomials, indexed by the set $\mathcal{P}_n$ of partitions of length $\leq n$.

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1.1 **Uniform notation**

A key observation in this paper is that the symmetric and nonsymmetric settings share a key combinatorial structure described in Section 2.1; in each case the index set is a locally finite graded poset with $\hat{0}$.

This enables us to formulate and prove our results in a unified manner. To this end, we introduce the following notation: we write L for the index set ($\mathcal{C}_n$ or $\mathcal{P}_n$) and $\mathcal{R}$ for the corresponding polynomial ring (nonsymmetric or symmetric). We denote the partial order on L by $\supseteq$, its covering relation by $:\supset$, and the rank of $r \in L$ by $|r|$. For r in L, we let $\overline{r} \in \mathbb{F}^n$ be as in formula (11) in Section 2.2.1.

The interpolation polynomial h_s, defined in [6, 7, 17, 19], is the unique polynomial of degree $|s|$ in $\mathcal{R}$ such that

$$h_s\left(\overline{r}\right) = \delta_{rs} \text{ for all } r \in L \text{ with } |r| \le |s|. \tag{1}$$

The "extra vanishing" theorem in [7, Theorem 5.2] and [6, Theorem 4.5] shows

$$h_s\left(\overline{r}\right) = 0 \text{ unless } r \supseteq s. \tag{2}$$

The Jack polynomial g_s is the homogeneous polynomial of degre $|s|$ defined up to a multiple by the formula

$$g_s = k_s\left[h_s\right]; \tag{3}$$

here $[h_s]$ is the top degree part of h_s, and k_s is a constant that we fix by requiring

$$g_s\left(\mathbf{1}\right) = 1 \text{ where } \mathbf{1} = (1, 1, \ldots, 1). \tag{4}$$

We note that the original definitions of the Jack polynomials in [4, 16] are quite different; formula (3) is the key result of [6, 7].

Although $\mathcal{P}_n$ is a subset of $\mathcal{C}_n$, the symmetric and nonsymmetric polynomials indexed by an element of $\mathcal{P}_n$ are in general quite different. However, this ambiguity will not be an issue because, while we treat the two cases in parallel, we do not consider formulas that *simultaneously* involve both symmetric and nonsymmetric Jack polynomials.

1.2 Translation

We first consider the operator of translation by $\mathbf{1}$ in $\mathcal{R}$, and write $B = (b_{rs})$ for its matrix with respect to the Jack basis $\{g_r : r \in L\}$ so that

$$g_r(x + \mathbf{1}) = \textstyle\sum_s b_{rs} g_s(x). \tag{5}$$

Here and elsewhere, we write $x = (x_1, \dots, x_n)$.
The coefficients b_{rs} are called binomial coefficients in [5, 11, 15, 20]. It was shown in [15, Theorem 3.2] and [20, Corollary 1.9] that

$$b_{rs} = h_s(\overline{r}). \tag{6}$$

By formulas (1), (2), and (6), we have $b_{ss} = 1$ and that

$$b_{rs} = 0 \text{ unless } r \supseteq s. \tag{7}$$

The coefficients b_{rs} for $r :\supset s$ have been computed explicitly in [5, 14], see Section 2.5.2 below. Our first result is a formula for the other binomial coefficients in terms of these. For this, we define a matrix $A = (a_{rs})$ as follows:

$$a_{rs} = \begin{cases} b_{rs} \text{ if } r :\supset s \\ 0 \quad \text{otherwise} \end{cases} \tag{8}$$

Theorem 1. We have $B = \exp(A)$. □

As shown in Section 3 below, this is equivalent to the following result.

Theorem 2. The coefficients b_{rs} satisfy the following recursions:

$$\textit{(i)}\ b_{rr} = 1, \ \textit{(ii)}\ (|r| - |s|)\, b_{rs} = \textstyle\sum_{u :\supset s} b_{ru} a_{us}. \qquad \square$$

1.3 Multiplication

We next consider the operator of multiplication by p in $\mathcal{R}$. Let $C = C(p)$ denote the *transpose* of its matrix with respect to the interpolation basis $\{h_r : r \in L\}$, so that we have

$$ph_s = \textstyle\sum_r c_{rs} h_r. \tag{9}$$

Following [21], we refer to $c_{rs} = c_{rs}(p)$ as Littlewood–Richardson (LR) coefficients. Define a diagonal matrix $D = D(p)$ as follows:

$$d_{rs} = \begin{cases} p(\overline{r}) \text{ if } r = s \\ 0 \quad \text{otherwise} \end{cases}. \tag{10}$$

Theorem 3. We have $C = B^{-1}DB$. □

As shown in Section 3, this is equivalent to the following:

Theorem 4. The coefficients c_{rs} satisfy the following recursions:

$$\text{(i) } c_{rr} = p(\overline{r}), \text{ (ii) } (|r| - |s|)\, c_{rs} = \textstyle\sum_{u:\supset s} c_{ru} a_{us} - \sum_{v \subset: r} a_{rv} c_{vs}.$$ □

1.4 Positivity

While the above results are the same in the symmetric and nonsymmetric settings, the two cases are quite different with respect to considerations of positivity. To explain this, we need additional notation. Let $\mathbb{F}_+$ denote the subset of $\mathbb{F} = \mathbb{Q}(\alpha)$ consisting of elements that can be written as a quotient of two polynomials in $\mathbb{N}[\alpha]$. Note that $\mathbb{F}_+$ is a convex multiplicative cone, that is, it is closed under addition, multiplication, and scalar multiplication by $\mathbb{Q}_+$. We also write $\mathbb{F}_{++}$for the open subcone consisting of elements that have a nonzero specialization at $\alpha = 0$ (and hence at any $a \in \mathbb{Q}_+$).

Theorem 5. In the symmetric case, for $r \supseteq s$ the binomial coefficients b_{rs} belong to $\mathbb{F}_{++}$. Moreover, for any $r, s \in L$ we have

$$b_{rs} = 0 \text{ for some } \alpha \in \mathbb{Q}_+ \iff b_{rs} \text{ is identically } 0 \iff r \not\supseteq s.$$ □

Proof. The remark after formula (13) shows that for $u :\supset v$ in the symmetric case, $a_{uv} \in \mathbb{F}_{++}$. Now suppose $r \supseteq s$ with $k = |r| - |s|$, then by Theorem 1

$$b_{rs} = \frac{1}{k!} \sum a_{u_0, u_1} a_{u_1, u_2} \ldots a_{u_{k-1}, u_k}$$

where the sum runs over all sequences $r = u_0 :\supset u_1 :\supset u_2 \cdots :\supset u_k = s$. This implies that for $r \supseteq s$ we have $b_{rs} \in \mathbb{F}_{++}$. The rest of the theorem follows from formula (2). ■

It seems that a similar phenomenon holds for the LR coefficients for symmetric interpolation polynomials, but we do not have a proof. Consider $p = h_t$ in the definition (9) of LR coefficients, and write $c_{r,st} = c_{rs}(h_t)$.

Conjecture 6. In the symmetric case, all the LR coefficients $c_{r,st}$ belong to $\mathbb{F}_+$. □

The nonsymmetric analogues of Theorem 5 and Conjecture 6 are false. For instance, if $r = [2, 1, 2]$ and $s = [1, 2, 1]$, then we have

$$b_{rs} = c_{r,rs} = -(1 + 2\alpha)/(\alpha + 2)(\alpha + 1)^2.$$

1.5 Remarks

In [21], Stanley considered the expansion of the product of two symmetric Jack polynomials in terms of symmetric Jack polynomials, and conjectured that the corresponding coefficients, after suitable normalization, belong to $\mathbb{N}[\alpha]$. While it follows from [19] that these coefficients are in $\mathbb{Z}[\alpha]$, their nonnegativity is as yet unproven and is perhaps the most important outstanding problem regarding these polynomials.

It follows from formula (3) that for $|r| = |s| + |t|$, Stanley's LR coefficients agree with the symmetric $c_{r,st}$ up to an explicit renormalization. It is easy to see that Stanley's conjecture implies Conjecture 6 for $|r| = |s| + |t|$.

The organization of this paper is follows: in the next section we recall known results about compositions, partitions, and Jack and interpolation polynomials. In Section 3, we establish the equivalence of Theorems 1 and 2, as well as that of Theorems 3 and 4. This section is written in the context of arbitrary locally finite graded posets [3]; the generality of the setting provides a substantial simplification, and the considerations here may have applications beyond the current setting. We use these results in Section 4 to prove the main theorems.

2 Preliminaries

2.1 Compositions and partitions

2.1.1

As before, we write $\mathcal{C}_n$ and $\mathcal{P}_n$, respectively for the sets of compositions and partitions with at most n parts. An element of $\mathcal{C}_n$ is simply an n-tuple of nonnegative integers

$\eta = (\eta_1, \ldots, \eta_n)$; the η_i are called the parts of η. Their sum $|\eta| = \eta_1 + \ldots + \eta_n$ is the rank of η. A partition is a composition λ that has nonincreasing parts

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

2.1.2

We recall from [6] the definition of the inclusion partial order $\supseteq$ on $\mathcal{C}_n$. For γ, η in $\mathcal{C}_n$, write $\eta :\supset \gamma$ if there are indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

$$\eta_i = \begin{cases} \gamma_{i_1} + 1 & \text{if } i = i_k \\ \gamma_{i_{j+1}} & \text{if } i = i_j,\ \ j < k \\ \gamma_{i_1} & \text{otherwise} \end{cases}.$$

The partial order $\supseteq$ is defined to be the reflexive and transitive closure of $:\supset$, and conversely $:\supset$ is the covering relation of $\supseteq$.

For λ, μ in $\mathcal{P}_n$ the relation $\lambda :\supset \mu$ forces $k = 1$ in the above definition. Thus, the restriction of $\supseteq$ to $\mathcal{P}_n$ is the usual inclusion order as defined in [13], with $\lambda \supseteq \mu$ if and only if $\lambda_i \geq \mu_i$ for all i.

2.1.3

The partial order $\supseteq$ and the rank function $|\cdot|$ provide each of the two sets $\mathcal{C}_n$ and $\mathcal{P}_n$ with the structure of a graded, locally finite, poset with unique minimum element $\hat{0} = (0, 0, \ldots, 0)$.

2.2 **Constants**

Once again, we let $\mathbb{F} = \mathbb{Q}(\alpha)$ be the field of rational functions in a parameter α. We collect in this subsection the definitions of various constants, associated to compositions and partitions, that are needed in the theory of Jack polynomials and interpolation polynomials.

2.2.1

The symmetric group S_n acts naturally on $\mathcal{C}_n$ and the S_n-orbit of each element η in $\mathcal{C}_n$ contains a unique partition that we denote η^+. As usual, we define the length of $w \in S_n$ to be the number of inversions of w, this is the number of pairs of indices (i, j) such that

$i < j$ and $w(i) > w(j)$. For η in $\mathcal{C}_n$, there is a unique shortest element w_η in S_n such than $\eta = w_\eta(\eta^+)$, and we define $\overline{\eta} \in \mathbb{F}^n$ by

$$\overline{\eta} = w_\eta\left(\eta^+ + \rho\right) = \eta + w_\eta \rho \tag{11}$$

where $\rho = (0, -1/\alpha, \ldots, -(n-1)/\alpha)$.

Note that for $\eta, \gamma \in \mathcal{C}_n$, we have

$$\textstyle\sum_i \overline{\eta}_i - \sum_i \overline{\gamma}_i = \left(\sum_i \eta_i + \sum_i \rho_i\right) - \left(\sum_i \gamma_i + \sum_i \rho_i\right) = |\eta| - |\gamma| \tag{12}$$

Also note that $\lambda \in \mathcal{P}_n$, we have $w_\lambda = id$ and $\overline{\lambda} = \lambda + \rho$.

2.2.2

The Young diagram of a composition η is a left-justified rectangular array of boxes, with η_i boxes in row i. Let $s = (i, j)$ denote the jth box in row i, and define the *arm* and *coarm* of s to be the number of boxes to its right and left:

$$\mathrm{arm}(s) := \eta_i - j, \ \mathrm{coarm}(s) = i - 1.$$

We also define the *leg* and *coleg* of s as follows:

$$\begin{aligned}\mathrm{leg}(s) &:= \#\{k > i : j \le \eta_k \le i\} + \#\{k < i : j \le \eta_k + 1 \le \eta_i\}\\ \mathrm{coleg}(s) &:= \#\{k > i : \eta_k > \eta_i\} + \#\{k < i : \eta_k \ge \eta_i\}.\end{aligned}$$

Note that if λ is a partition then $\mathrm{leg}(s)$ and $\mathrm{coleg}(s)$ are the numbers of boxes directly below and directly above s, respectively.

2.2.3

For a box s in a composition η, we define

$$\begin{aligned}e_\eta(s) &= \alpha\,(\mathrm{coarm}(s) + 1) + n - \mathrm{coleg}(s)\\ d_\eta(s) &= \alpha\,(\mathrm{arm}(s) + 1) + (\mathrm{leg}(s) + 1)\\ d'_\eta(s) &= \alpha\,(\mathrm{arm}(s) + 1) + \mathrm{leg}(s)\end{aligned}$$

and we put $e_\eta = \prod_{s \in \eta} e_\eta(s)$, $d_\eta = \prod_{s\in\eta} d_\eta(s)$, $d'_\eta = \prod_{s\in\eta} d'_\eta(s)$, $f_\eta = d_\eta d'_\eta$.

For a box s in a partition λ, we define

$$b_\lambda(s) = \alpha \mathrm{coarm}(s) + n - \mathrm{coleg}(s)$$
$$c_\lambda(s) = \alpha \mathrm{arm}(s) + (\mathrm{leg}(s) + 1)$$
$$c'_\lambda(s) = \alpha\,(\mathrm{arm}(s) + 1) + \mathrm{leg}(s)$$

and we put $b_\lambda = \prod_{s\in\lambda} b_\lambda(s)$, $c_\lambda = \prod_{s\in\lambda} c_\lambda(s)$, $c'_\lambda = \prod_{s\in\lambda} c'_\lambda(s)$, $j_\lambda = c_\lambda c'_\lambda$.

2.3 Interpolation polynomials

2.3.1

Symmetric interpolation polynomials were first introduced by the author in [17] in connection with a joint work with B. Kostant [9, 10] on the Capelli identity. They are characterized as follows:

Proposition 7. For each $\lambda \in \mathcal{P}_n$, there is a unique symmetric polynomial $R_\lambda = R_\lambda^{(\alpha)}(x)$ in $\mathcal{S}$ such that $\deg(R_\lambda) \leq |\lambda|$ and

$$R_\lambda(\overline{\mu}) = \delta_{\lambda\mu} \text{ if } \mu \in \mathcal{P}_n, |\mu| \leq |\lambda|. \qquad \square$$

It is shown in [7, Theorem 5.2] that $R_\lambda(\overline{\mu}) = 0$ unless $\mu \supseteq \lambda$.

2.3.2

Nonsymmetric interpolation polynomials were introduced by the author and F. Knop in [6, 19]. They are characterized as follows:

Proposition 8. For each composition $\eta \in \mathcal{C}_n$, there is a unique polynomial $G_\eta = G_\eta^{(\alpha)}(x)$ in $\mathcal{R}$ such that $\deg(G_\eta) \leq |\eta|$ and

$$G_\eta(\overline{\gamma}) = \delta_{\eta\gamma} \text{ if } \gamma \in \mathcal{C}_n, |\gamma| \leq |\eta|. \qquad \square$$

It is shown in [6, Theorem 4.5] that $G_\eta(\overline{\gamma}) = 0$ unless $\gamma \supseteq \eta$.

2.4 **Jack polynomials**

2.4.1

Symmetric Jack polynomials $J_\lambda = J_\lambda^{(\alpha)}(x)$ were introduced by H. Jack [4] as a common generalization of Schur polynomials and zonal polynomials, which are spherical polynomials for $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$, respectively. They were further studied in [8, 12, 13, 21] where some of their key properties were established. Their connection with the interpolation polynomials was proved by F. Knop and the author in [7]:

$$R_\lambda(x) = \frac{1}{j_\lambda} J_\lambda + \text{lower degree terms.}$$

One also has the evaluation formula $J_\lambda(\mathbf{1}) = b_\lambda$.

2.4.2

Nonsymmetric Jack polynomials $F_\eta = F_\eta^{(\alpha)}(x)$ were introduced in [16] of as eigenfunctions of Cherednik operators [2] for general root systems, and studied further in [8]. The connection with nonsymmetric interpolation polynomials was proved in [6]:

$$G_\eta = \frac{1}{f_\eta} F_\eta + \text{lower degree terms.}$$

One also has the evaluation formula $F_\eta(\mathbf{1}) = e_\eta$.

2.4.3

As a consequence of the above discussion, we can give an explicit formula for the constant k_s in formula (3). We have

$$k_s = \begin{cases} b_\lambda / j_\lambda & \text{if } s = \lambda \text{ in the symmetric case} \\ e_\eta / f_\eta & \text{if } s = \eta \text{ in the nonsymmetric case.} \end{cases}$$

2.5 The binomial formula

2.5.1

A further connection of the interpolation polynomials with Jack polynomials in the symmetric case was discovered by A. Okounkov and G. Olshanski [15, Theorem 3.2], who proved the following binomial formula (see also [5, 11] for earlier work):

$$\frac{J_\lambda(x+\mathbf{1})}{J_\lambda(\mathbf{1})} = \sum\nolimits_\mu R_\mu(\overline{\lambda}) \frac{J_\mu(x)}{J_\mu(\mathbf{1})}.$$

The analogous formula for nonsymmetric polynomials was proved in [20, Corollary 1.9] and independently in [1]

$$\frac{F_\eta(x+\mathbf{1})}{F_\eta(\mathbf{1})} = \sum\nolimits_\gamma G_\gamma(\overline{\eta}) \frac{F_\gamma(x)}{F_\gamma(\mathbf{1})}.$$

The special values $R_\mu(\overline{\lambda})$ and $G_\gamma(\overline{\eta})$ are called symmetric and nonsymmetric binomial coefficients, respectively. For $n=1$, they reduce to the usual binomial coefficients.

2.5.2

For $\lambda :\supset \mu$, there is an explicit formula for $R_\mu(\overline{\lambda})$, first proved in [5, Proposition 2]. In this case, λ and μ differ by a single box s_0, and we let C and R denote the other boxes in the column and row of s_0, respectively. Then we have

$$R_\mu(\overline{\lambda}) = \left(\prod_{s\in C} \frac{c_\lambda(s)}{c_\mu(s)}\right)\left(\prod_{s\in R} \frac{c'_\lambda(s)}{c'_\mu(s)}\right). \tag{13}$$

From the formulas in Section 2.2.3, it follows that the ratios $c_\lambda(s)/c_\mu(s)$ and $c'_\lambda(s)/c'_\mu(s)$ belong to $\mathbb{F}_+$. A slightly more careful analysis shows that they belong to $\mathbb{F}_{++}$. This is obvious for $c_\lambda(s)/c_\mu(s)$ since the numerator and denominator are nonzero for the specialization $\alpha=0$. Now it is possible that $c'_\lambda(s)$is a multiple of α, if $\mathrm{leg}_\lambda(s)=0$, but in this case $\mathrm{leg}_\mu(s) = \mathrm{leg}_\lambda(s)=0$ as well and so $c'_\lambda(s)/c'_\mu(s)$is a constant. Thus, we deduce that $R_\mu(\overline{\lambda}) \in \mathbb{F}_{++}$ for $\lambda :\supset \mu$.

In the nonsymmetric case, the analogous formula for $G_\gamma(\overline{\eta})$ with $\eta :\supset \gamma$ was first obtained in [14, Corollary 4.2]. Suppose $\eta :\supset \gamma$ and let $1 \le i_1 < i_2 < \ldots < i_k \le n$ be the corresponding indices as in 2.1.2. Then we have

$$G_\gamma(\overline{\eta}) = -[\alpha a_n + n - 1] \cdot \prod_{j=1}^{n} \frac{b_j - \overline{\gamma}_j}{a_j - \overline{\gamma}_j}$$

where

$$a_j = \begin{cases} \overline{\gamma}_{i_l} & j \in [i_{l-1}, i_l) \\ \overline{\gamma}_{i_1} + 1 & j \ge i_k \end{cases}, \quad b_j = \begin{cases} \overline{\gamma}_{i_l} - 1/\alpha & j \in (i_{l-1}, i_l] \\ \overline{\gamma}_{i_1} - 1/\alpha & j > i_k \end{cases}.$$

3 Graded Posets

For this section only, we let $(L, \ge)$ denote an *arbitrary* locally finite graded poset with a unique minimal element $\hat{0}$. Local finiteness means that all the intervals $[s, r] := \{x \in L : r \ge x \ge s\}$ are finite sets. We write $:>$ for the associated covering relation; thus $r :> s$ means that $r \ge s$ and $[s, r] = \{s, r\}$. We also assume that the grading $r \mapsto |r|$ satisfies the usual properties, namely

$$\left|\hat{0}\right| = 0 \quad \text{and} \quad r :> s \Rightarrow |r| = |s| + 1. \tag{14}$$

For more background on such posets, we refer the reader to [3].

3.1 Incidence algebra

Also in this section only, we let $\mathbb{F}$ denote an arbitrary field. The incidence algebra of L is the algebra $\mathcal{A}$ of $L \times L$ matrices M with entries in $\mathbb{F}$, satisfying $m_{rs} = 0$ unless $r \ge s$. The local finiteness of L ensures that the product of two matrices in $\mathcal{A}$ is well defined, and is in $\mathcal{A}$, so that $\mathcal{A}$ is indeed an associative algebra. Moreover, for $M \in \mathcal{A}$, the matrix exponential

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!} \tag{15}$$

is a well-defined element of $\mathcal{A}$. To see this we note that the rsth entry of $\exp(M)$ is the same as that in the exponential of the *finite* $[s, r] \times [s, r]$ submatrix.

3.2 Binomial coefficients

The first result is a recognition theorem for binomial coefficients.

Theorem 9. Let B be an arbitrary $L \times L$ matrix, and define the matrix A by

$$a_{rs} = \begin{cases} b_{rs} & \text{if } r :> s \\ 0 & \text{otherwise} \end{cases}.$$

Then the following are equivalent

(a) $B = \exp(A)$, and in particular B is in $\mathcal{A}$,
(b) *(i)* $b_{rr} = 1$ for all r, *(ii)* $(|r| - |s|)\, b_{rs} = \sum_{u:>s} b_{ru} a_{us}$ for all r, s,
(c) *(i)* $b_{rr} = 1$ for all r, *(ii)* $(|r| - |s|)\, b_{rs} = \sum_{v<:r} a_{rv} b_{vs}$ for all r, s. □

Proof. Assume $B = \exp(A)$. Since $a_{rs} = 0$ unless $|r| - |s| = 1$, (15) implies

$$b_{rs} = \begin{cases} \left(\frac{A^n}{n!}\right)_{rs} & \text{if } |r| - |s| = n \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

Thus, $b_{rr} = 1$ for all r, and $(|r| - |s|)b_{rs}$ is the rsth entry of the matrix

$$\sum_{n=0}^{\infty} n \frac{A^n}{n!} = \sum_{n=1}^{\infty} \frac{A^n}{(n-1)!} = A \exp(A) = AB = BA.$$

Hence (a) implies (b) and (c). Conversely each of (b) and (c) *characterizes* b_{rs} by induction on $||r| - |s||$; thus (b) and (c) each imply (a). ■

Corollary 10. If B satisfies the conditions of Theorem 9 then B is invertible, and the rsth entry of B^{-1} is $(-1)^{|r|-|s|}\, b_{rs}$. □

Proof. This follows from (16) since $B^{-1} = \exp(-A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A^n$. ■

3.3 Littlewood-Richardson coefficients

The next proposition gives a characterization of LR coefficients.

Theorem 11. Let A and B satisfy the three equivalent conditions of Theorem 9, fix a diagonal matrix D, and let C be an arbitrary $L \times L$ matrix. Then the following are equivalent

(a) $d_{uu}b_{us} = \sum_r c_{rs} b_{ur}$ for all u in L,

(b) $DB = BC$,

(c) $C = B^{-1}DB$, and in particular C is in $\mathcal{A}$,

(d) $c_{rs} = \sum_u (-1)^{|r|-|u|}\, b_{ru} d_{uu} b_{us}$,

(e) *(i)* $c_{rr} = d_{rr}$, *(ii)* $(|r| - |s|)\, c_{rs} = \sum_{u:>s} c_{ru} a_{us} - \sum_{v<:r} a_{rv} c_{vs}$ for all r, s. □

Proof. Statements (a), (b), and (c) are trivially equivalent and (d) is equivalent to (c) by the previous corollary. Suppose now that (d) holds. Since B is in $\mathcal{A}$, the only possibly nonzero summands in (d) are those for which $r \geq u \geq s$. For c_{rr}, only the $u = r$ term survives, and (d) implies (e) (i) as follows:

$$c_{rr} = (-1)^{|r|-|r|}\, b_{rr} d_{rr} b_{rr} = d_{rr}.$$

Using Theorem 9 and (14), we obtain (e)(ii) from (d) as follows:

$$\begin{aligned}(|r| - |s|) c_{rs} &= \textstyle\sum_t (|r| - |t| + |t| - |s|)\, (-1)^{|r|-|t|}\, b_{rt} d_{tt} b_{ts} \\ &= \textstyle\sum_t (-1)^{|r|-|t|} \left[\sum_{u<:r} a_{ru} b_{ut}\right] d_{tt} b_{ts} \\ &\quad + \textstyle\sum_t (-1)^{|r|-|t|}\, b_{rt} d_{tt} \left[\sum_{v:>s} b_{tv} a_{vs}\right] \\ &= -\textstyle\sum_{u<:r} a_{ru} c_{us} + \sum_{v:>s} c_{rv} a_{vs}.\end{aligned}$$

Conversely (e) characterizes c_{rs} by induction on $||r| - |s||$; so (e) implies (d). ■

4 Proofs of the Main Results

4.1

We first prove Theorems 1, 2, and 5. In the symmetric case, Theorem 2 was first proved in [5] and our proof is a generalization of the argument in [15].

Proof. By formula (6), we get $b_{rr} = h_r(\overline{r}) = 1$, which is part (i) of Theorem 2. Also by formulas (6) and (12), part (ii) of Theorem 2 is the specialization at $x = \overline{r}$ of the following polynomial identity:

$$\left(\sum_i x_i - \sum_i \overline{s}_i\right) h_s(x) = \sum_{u:\supset s} a_{us} h_u(x). \tag{17}$$

To *prove* (17), we let ϕ denote the left side. We claim that ϕ vanishes if $x = \overline{r}$ with $|r| \leq |s|$. Indeed if $|r| < |s|$ then $h_s(\overline{r}) = 0$, while if $|r| = |s|$ by formula (12) we get $\sum_i \overline{r}_i - \sum_i \overline{s}_i = |r| - |s| = 0$. Since ϕ is a polynomial of degree $|s| + 1$, the vanishing conditions and formulas (1–2) imply that

$$\left(\sum_i x_i - \sum_i \overline{s}_i\right) h_s(x) = \sum_{u:\supset s} k_{us} h_u(x)$$

for some coefficients k_{us}. Fix $v :\supset s$ and substitute $x = \overline{v}$ in the above to get

$$(|v| - |s|)\, h_s(\overline{v}) = \sum_{u:\supset s} k_{us}\delta_{uv} \implies a_{vs} = k_{vs}.$$

Theorem 1 follows from Theorems 2 and 9. ■

4.2

We now prove Theorems 3 and 4.

Proof. By the definition (9) of C, we have

$$\sum_r c_{rs} h_r(x) = p(x)\, h_s(x)\,.$$

Evaluating at $x = \overline{u}$, we get

$$p(\overline{u})\, h_s(\overline{u}) = \sum_r h_r(\overline{u})\, c_{rs}$$

and recalling the definitions of B (5) and D (10) this implies

$$d_{uu} b_{us} = \sum_r b_{ur} c_{rs}.$$

Theorems 3 and 4 now follow from Theorems 11 and 2. ■

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