



## Convolution of Invariant Distributions: Proof of the Kashiwara–Vergne conjecture

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**Abstract.** Consider the Kontsevich star-product on the symmetric algebra of a finite-dimensional Lie algebra  $\mathfrak{g}$ , regarded as the algebra of distributions with support 0 on  $\mathfrak{g}$ . In this Letter, we extend this star product to distributions satisfying an appropriate support condition. As a consequence, we prove a long-standing conjecture of Kashiwara–Vergne on the convolution of germs of invariant distributions on the Lie group  $G$ .

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### 0. Introduction

In several problems in harmonic analysis on Lie groups, one needs to relate invariant distributions on a Lie group  $G$  to invariant distributions on its Lie algebra  $\mathfrak{g}$ . For instance, it is a central aspect in Harish-Chandra's work in the semi-simple case. The symmetric algebra  $\mathcal{S}(\mathfrak{g})$  and the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  can be regarded as convolution algebras of distributions supported at 0 in  $\mathfrak{g}$  and 1 in  $G$ , respectively. Therefore the Harish-Chandra isomorphism between the ring of invariants in  $\mathcal{S}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$  can be seen in this light. At a more profound level, Harish-Chandra's regularity result for invariant eigendistributions on the group involves the lifting of the corresponding result on the Lie algebra.

Using the orbit method of Kirillov, Duflo defined an isomorphism extending the Harish-Chandra homomorphism to the case of general Lie groups [6–8]. This result was crucial in the proof of local solvability of invariant differential operators on Lie groups, established by Raïs [19] for nilpotent groups, then by Duflo and Raïs [9] for solvable groups, and finally by Duflo [8] in the general case.

Soon thereafter, Kashiwara and Vergne [12] conjectured that a natural extension of the Duflo isomorphism to germs of distributions on  $\mathfrak{g}$ , when restricted to invariant germs with appropriate support, should carry the convolution on  $\mathfrak{g}$  to the convolution on  $G$ . (See also an observation of Raïs in [20].)

In this Letter, we prove the Kashiwara–Vergne conjecture in full generality, using, as in [3], the Kontsevich quantization of the dual of a Lie algebra.

### 0.1. A SHORT REVIEW OF THE SUBJECT

In their paper [12], Kashiwara and Vergne propose a combinatorial conjecture on the Campbell–Hausdorff formula and prove that it implies the conjecture on distributions. In that same paper, they prove the combinatorial conjecture for solvable groups.

Apart from the solvable case and the case of  $\mathfrak{sl}(2, \mathbb{R})$  considered by Rouvière [21], the Kashiwara–Vergne (KV) conjectures resisted all attempts until 1999. Then the conjecture on distributions (henceforth KVR conjecture) was established for arbitrary groups, but under the restriction that one of the distributions have point support [2, 3]. This suffices for many applications, including the local solvability result mentioned above. Shortly thereafter, Vergne [30] proved the combinatorial conjecture, but for a special class of Lie algebras, the *quadratic* Lie algebras (those admitting an invariant nondegenerate quadratic form).

A first version of this Letter, not substantially different from this one, was written in 2001 [4]. In the interval, several related papers have appeared.

Let us mention, in the spirit of Kontsevich’s original approach, [16] which proves the existence of a  $A_\infty$ -tangent quasi-isomorphism for the Kontsevich formality theorem and deduces a proof of the KVR conjecture (see also [15]); [10] which constructs a  $G_\infty$ -tangent quasi-isomorphism as in [26], generalizing the  $A_\infty$  and  $L_\infty$  cases (formulas in [10] are not explicit in as in [16]).

The paper [27] sheds some light on the deformation quantization approach to the combinatorial KV conjecture. Indeed, using the homotopy argument from [14] which is crucial in [3] and here, [27] constructs a two-dimensional deformation (with the corresponding differential equation) for the Campbell–Hausdorff formula. That deformation differs from the one considered in [12] which leads to combinatorial KV, but it implies KVR just as combinatorial KV does.

Two papers [28, 29] extend [27] to the case of symmetric spaces. The paper [18] (see also [25]) extends the Duflo isomorphism to all cohomology spaces.

In 2002, using Poisson geometry, Alekseev and Meinrenken [1] obtained a conceptual proof of combinatorial KV in the case of a quadratic algebra, i.e. the same result as [30]. But the combinatorial conjecture remains unproven in general.

Since this Letter deals with the more analytic aspects of the theory (distributions, germs, symbols) and addresses somewhat subtle convergence problems, it remains of independent interest.

0.2. OUTLINE OF THE LETTER

Let us now outline the result and our method. Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ , and  $\exp: \mathfrak{g} \rightarrow G$  the exponential map. Let  $q$  be the analytic function on  $\mathfrak{g}$  defined by

$$q(X) = \det \left( \frac{e^{\text{ad } X/2} - e^{-\text{ad } X/2}}{\text{ad } X} \right)^{1/2}. \tag{1}$$

Define, for a distribution  $u$  on  $\mathfrak{g}$

$$\eta(u) = \exp_*(u.q), \tag{2}$$

where  $\exp_*$  is the pushforward of distributions under the exponential map. We consider germs at 0 of distributions on  $\mathfrak{g}$ , henceforth simply *germs*. For distributions  $u, v, \dots$  the corresponding germs will be denoted  $u, v, \dots$ ; we use the same notation  $\eta$  for the induced map on germs.

We wish to consider the convolution of germs. This notion is well defined under a certain asymptotic support condition which we now describe. If  $U$  is a subset of  $\mathbb{R}^n$ , we define the asymptotic cone of  $U$  at  $x \in \mathbb{R}^n$  as the set  $C_x(U)$  of limit points of all sequences

$$a_n(x_n - x), \tag{3}$$

for all sequences  $x_n \in U, x_n \rightarrow x$  and all sequences  $a_n \in \mathbb{R}^+$ . Clearly,  $C_x(U)$  depends only on the intersection of  $U$  with an arbitrarily small neighborhood of  $x$ . If  $M$  is a manifold, and  $x \in M, U \subset M$ , using a coordinate chart, we can once again define  $C_x(U)$  as a cone in the tangent space  $T_x M$ .

Let  $u$  be a distribution on  $\mathfrak{g}$ . Then  $C_0(\text{supp } u)$  depends only on the germ  $u$ , and we will write it as  $C_0[u]$ . Assume that  $u, v$  are germs such that

$$C_0[u] \cap -C_0[v] = \{0\}. \tag{4}$$

When two germs verify (4) we will say that they are *compatible*. In this case the (Abelian) convolution on the Lie algebra  $u *_{\mathfrak{g}} v$  is well defined as a germ on  $\mathfrak{g}$ . Also, using the Campbell–Hausdorff formula, it is easy to see that the Lie group convolution  $\eta(u) *_{G} \eta(v)$  is well defined as the germ at 1 of a distribution on the group  $G$ .

The Lie algebra  $\mathfrak{g}$  acts on functions on  $\mathfrak{g}$  by adjoint vector fields. The dual action descends to germs. We call a germ *invariant* if it is annihilated by all elements of the Lie algebra. Our main theorem is

**THEOREM 0.1** (Kashiwara–Vergne–Raïs conjecture). *Assume that  $u$  and  $v$  are compatible invariant germs. Then*

$$\eta(u *_{\mathfrak{g}} v) = \eta(u) *_{G} \eta(v). \tag{5}$$

As in [12], one can reformulate the conjecture slightly by considering, for  $t \in \mathbb{R}$ , the Lie group  $G_t$  with Lie algebra  $\mathfrak{g}_t$ , where  $\mathfrak{g}_t$  is  $\mathfrak{g}$  as a vector space, equipped with the Lie bracket

$$[X, Y]_t = t[X, Y]. \quad (6)$$

The function  $q(X)$  must then be changed to  $q_t(X) = q(tX)$  and, accordingly,  $\eta$  to  $\eta_t$ . Let  $u$  and  $v$  be compatible germs on  $\mathfrak{g}$ ,  $u$  and  $v$  distributions with small compact support representing  $u$  and  $v$ , and  $\phi$  a smooth function; we can define

$$\Psi(t) = \Psi_{u,v,\phi}(t) = \langle \eta_t^{-1}(\eta_t(u) *_{G_t} \eta_t(v)), \phi \rangle \quad (7)$$

as a function of  $t \in \mathbb{R}$ . Clearly

$$\Psi(0) = \langle u *_{\mathfrak{g}} v, \phi \rangle \quad \text{and} \quad \Psi(1) = \langle \eta^{-1}(\eta(u) *_{G} \eta(v)), \phi \rangle.$$

The Kashiwara–Vergne conjecture is implied by (and in fact equivalent to) the statement that for  $u$  and  $v$  *invariant* germs, and for all  $\phi$  with sufficiently small support (depending on  $u$  and  $v$ ), the function  $\Psi_{u,v,\phi}$  is constant.

Using the Campbell–Hausdorff formula, it can be verified that  $\Psi$  is a differentiable function of  $t$ . Thus, it suffices to show that for  $u$  and  $v$  invariant

$$\Psi'(t) = 0 \text{ for all } t. \quad (8)$$

This precisely the approach of Kashiwara and Vergne in [12]: the combinatorial conjecture implies the vanishing of  $\Psi'(t)$ .

Our approach is different. We first show that if  $u$  and  $v$  are invariant, then  $\Psi(t)$  is *analytic* in  $t$ . Thus it suffices to prove that

$$\Psi^{(n)}(0) = 0 \text{ for all } n. \quad (9)$$

While at first this may not seem to be an easy problem, in this Letter we relate the group convolution to an extension of the Kontsevich star product to distributions and prove an equivalent statement concerning the star product.

We now recall the construction of Kontsevich. In [14], an associative star product is defined on any Poisson manifold, given by a formal series in a parameter  $\hbar$

$$u \star_{\hbar} v = \sum \frac{\hbar^n}{n!} B_n(u, v), \quad (10)$$

where  $u, v$  are  $C^\infty$ -functions on the manifold, and  $B_n(u, v)$  are certain bi-differential operators.

Consider  $\mathfrak{g}^*$ , the dual of  $\mathfrak{g}$ , equipped with its natural Poisson structure. It is easy to see that when  $u, v$  are in  $\mathcal{S}(\mathfrak{g})$ , i.e. polynomial functions on  $\mathfrak{g}^*$ , the formula for  $u \star_{\hbar} v$  is locally finite, so that one can set  $\hbar = 1$ , and then  $u \star_1 v$  is again in  $\mathcal{S}(\mathfrak{g})$ . Now regarding  $u$  and  $v$  as distributions supported at 0 on  $\mathfrak{g}$ ,  $\star_1$  can be considered as a new convolution on  $\mathfrak{g}$ , but defined *only* for distributions with point support at 0.

The star product is closely related to the multiplication in the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$ , which in turn is simply the group convolution  $*_G$  for distributions supported at 1 in  $G$ . Indeed, by the universal property of  $\mathcal{U}(\mathfrak{g})$ , there exists an isomorphism between  $(\mathcal{S}(\mathfrak{g}), \star_1)$  and  $(\mathcal{U}(\mathfrak{g}), *_G)$ . Kontsevich showed that this isomorphism is given explicitly in the form

$$u \in \mathcal{S}(\mathfrak{g}) \mapsto \eta(u\tau^{-1}), \tag{11}$$

where  $\tau$  was defined in [14] as a formal power series  $S_1(X)$ , and was shown in [3] to be an analytic function in a neighborhood of 0 in  $\mathfrak{g}$ . As it turns out, Shoikhet in [24] proves that in fact  $\tau \equiv 1!$  Taking this striking simplification into account is the main difference between [4] and the present paper – although it does not change the argument in any essential way.

More generally, setting  $\hbar$  equal to a real number  $t$  we deduce that

$$u \mapsto \eta_t(u) \tag{12}$$

is an isomorphism from  $(\mathcal{S}(\mathfrak{g}), \star_t)$  to  $(\mathcal{U}(\mathfrak{g}), *_G)$ . This implies the identity

$$\eta_t^{-1}(\eta_t(u) *_G \eta_t(v)) = u \star_t v, \tag{13}$$

for  $u, v$  distributions on  $\mathfrak{g}$  supported at 0.

Our first main result, proved in Section 3, is the following:

**THEOREM 0.2.** *The Kontsevich star product on  $\mathcal{S}(\mathfrak{g})$  extends to a (convolution) product*

$$(u, v) \mapsto u \star_t v, \tag{14}$$

for  $u$  and  $v$  distributions on  $\mathfrak{g}$  with sufficiently small support near 0. Moreover, formula (13) continues to hold.

In Section 4, we prove:

**THEOREM 0.3.** *The extended star product descends to a product on compatible germs also denoted  $\star_t$ . If  $u$  and  $v$  are compatible invariant germs, then  $u \star_t v$  is invariant. Furthermore*

$$u \star_t v = u *_G v. \tag{15}$$

The proof of (15) requires the analyticity of  $\Psi(t)$  together with an extension of the Kontsevich homotopy argument from [3].

Clearly, Theorems 0.3 and 0.2 imply Theorem 0.1.

### 1. Preliminaries

Let  $G$  be a finite-dimensional real Lie group with Lie algebra  $(\mathfrak{g}, [ \ , \ ])$  and fix a basis  $(e_i)_{1 \leq i \leq d}$  of  $\mathfrak{g}$ .

## 1.1. SYMMETRIC ALGEBRAS

Here, we work with the Lie algebra  $\mathfrak{g}$  and its dual, but they are considered as vector spaces. The symmetric algebra  $\mathcal{S}(\mathfrak{g})$  can be considered in three different ways:

- as the algebra  $\mathbb{R}[\mathfrak{g}^*]$  of polynomial functions on  $\mathfrak{g}^*$ ;
- as the algebra of constant coefficient differential operators on  $\mathfrak{g}$ : if  $p \in \mathcal{S}(\mathfrak{g})$ , we write  $\partial_p$  for the corresponding differential operator. For example, for  $p \in \mathfrak{g}$ ,  $\partial_p$  is the constant vector field defined by  $p$ ;
- finally, as the algebra (for convolution) of distributions on  $\mathfrak{g}$  with support  $\{0\}$  by the map  $p \mapsto \mathbf{d}_p$  where  $\langle \mathbf{d}_p, \phi \rangle = \partial_p(\phi)(0)$  for  $\phi$  a test function on  $\mathfrak{g}$ . (Up to some powers of  $i$ ,  $\mathbf{d}$  coincides with the Fourier transform from functions on  $\mathfrak{g}^*$  to distributions on  $\mathfrak{g}$ .)

When there is no ambiguity, we drop  $\mathbf{d}$ , and use the same notation for an element of  $\mathcal{S}(\mathfrak{g})$  as a polynomial function on  $\mathfrak{g}^*$  and as a distribution on  $\mathfrak{g}$ . We will then use  $\cdot$  or  $*_{\mathfrak{g}}$  for the product in  $\mathcal{S}(\mathfrak{g})$  depending on how we view elements of  $\mathcal{S}(\mathfrak{g})$ .

The symmetric algebra  $\mathcal{S}(\mathfrak{g}^*)$  has similar interpretations.

## 1.2. ALGEBRA OF DIFFERENTIAL OPERATORS

Let  $\mathfrak{W}(\mathfrak{g})$  the Weyl algebra of differential operators with polynomial coefficients on  $\mathfrak{g}$ . Any element in  $\mathfrak{W}(\mathfrak{g})$  can be uniquely written as a sum  $\sum q_i \partial_{p_i}$  with  $q_i \in \mathcal{S}(\mathfrak{g}^*)$  and  $p_i \in \mathcal{S}(\mathfrak{g})$ . In other words, we have a vector space isomorphism from  $\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})$  to  $\mathfrak{W}(\mathfrak{g})$ . The inverse map associates to a differential operator in  $\mathfrak{W}(\mathfrak{g})$  its symbol in  $\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})$ ; the symbol can be viewed as a polynomial map from  $\mathfrak{g}^*$  to  $\mathcal{S}(\mathfrak{g}^*)$ . We observe that an element of  $\mathfrak{W}(\mathfrak{g})$  is completely determined by its action on  $\mathcal{S}(\mathfrak{g}^*)$ .

Similarly, any element in  $\mathfrak{W}(\mathfrak{g}^*)$  can be written as a sum  $\sum_i p_i \partial_{q_i}$  with  $p_i \in \mathcal{S}(\mathfrak{g})$  and  $q_i \in \mathcal{S}(\mathfrak{g}^*)$ .

By duality with test functions, the set of distributions  $\mathbf{D}(\mathfrak{g})$  on  $\mathfrak{g}$  is a right  $\mathfrak{W}(\mathfrak{g})$ -module:  $\langle D \cdot L, \phi \rangle = \langle D, L \cdot \phi \rangle$ , where  $\phi \in C_c^\infty(\mathfrak{g})$ ,  $D \in \mathbf{D}(\mathfrak{g})$ ,  $L \in \mathfrak{W}(\mathfrak{g})$ . We can define a canonical anti-isomorphism (the *Fourier transform*)  $\mathcal{F}$  from  $\mathfrak{W}(\mathfrak{g}^*)$  to  $\mathfrak{W}(\mathfrak{g})$  such that  $\mathcal{F}(\sum p_i \partial_{q_i}) = \sum q_i \partial_{p_i}$  for  $p_i \in \mathcal{S}(\mathfrak{g})$  and  $q_i \in \mathcal{S}(\mathfrak{g}^*)$ . It verifies

$$\mathbf{d}_{L \cdot p} = \mathbf{d}_p \cdot \mathcal{F}(L) \tag{16}$$

for any  $L \in \mathfrak{W}(\mathfrak{g}^*)$ ,  $p \in \mathcal{S}(\mathfrak{g})$ .

1.3. MULTI-DIFFERENTIAL OPERATORS

Let  $\mathfrak{W}^m(\mathfrak{g}^*)$  be the set of  $m$ -differential operators with polynomial coefficients on  $\mathfrak{g}^*$ . These are linear combinations of operators from  $C^\infty(\mathfrak{g}^*)^{\otimes m}$  to  $C^\infty(\mathfrak{g}^*)$  of the form

$$(f_1 \otimes \cdots \otimes f_m) \mapsto p \partial_{q_1}(f_1) \cdots \partial_{q_m}(f_m), \tag{17}$$

where  $p \in \mathcal{S}(\mathfrak{g})$  and  $q_i \in \mathcal{S}(\mathfrak{g}^*)$ . There is an obvious linear isomorphism, written  $A \in \mathcal{S}(\mathfrak{g}) \otimes \bigotimes^m \mathcal{S}(\mathfrak{g}^*) \mapsto \partial_A \in \mathfrak{W}^m(\mathfrak{g}^*)$ . Its inverse maps a  $m$ -differential operator  $B$  to its *symbol*  $\sigma_B$ . A symbol will often be viewed as a polynomial map from  $\mathfrak{g}^m$  to  $\mathcal{S}(\mathfrak{g})$

For multidifferential operators there is no symmetry similar to the one given by the Fourier transform  $\mathcal{F}$  from  $\mathfrak{W}(\mathfrak{g}^*)$  to  $\mathfrak{W}(\mathfrak{g})$ . Nevertheless, in the case of bi-differential operators, for any  $B = p \partial_{q_1} \otimes \partial_{q_2} \in \mathfrak{W}^2(\mathfrak{g}^*)$ , we define an operator  $\mathcal{F}(B)$  mapping functions on  $\mathfrak{g}$  to functions on  $\mathfrak{g} \times \mathfrak{g}$ :

$$\mathcal{F}(B)(f)(x, y) = q_1(x) q_2(y) [\partial_p(f)](x + y). \tag{18}$$

By duality, we get a ‘right’ action on pairs of distributions on  $\mathfrak{g}$  in the following way:

$$(u, v) \cdot \mathcal{F}(B) = [(u \cdot q_1) *_{\mathfrak{g}} (v \cdot q_2)] \cdot \partial_p, \tag{19}$$

where  $u, v$  are distributions on  $\mathfrak{g}$ ,  $*_{\mathfrak{g}}$  is the convolution on  $\mathfrak{g}$  and it is assumed that the convolution makes sense. We then have a formula similar to (16)

$$\mathbf{d}_{B(f,g)} = (\mathbf{d}_f, \mathbf{d}_g) \cdot \mathcal{F}(B). \tag{20}$$

Note that a bi-differential operator with polynomial coefficients is completely determined by its action on point distributions.

1.4. ENVELOPING ALGEBRA

The enveloping algebra  $\mathcal{U} = \mathcal{U}(\mathfrak{g})$  of  $\mathfrak{g}$  can be seen as the algebra of *left* invariant differential operators on  $G$  (multiplication being composition of differential operators), as the algebra of distributions on  $G$  with support 1, multiplication being convolution of distributions. Depending on the situation, we will write  $\cdot$  or  $*_G$  the product in  $\mathcal{U}(\mathfrak{g})$ .

It is well known, and can be easily seen, that the symmetrization map  $\beta$  from  $\mathcal{S}(\mathfrak{g})$  to  $\mathcal{U}(\mathfrak{g})$  can be interpreted as the pushforward of distributions from the Lie algebra to the Lie group by the exponential map: for all  $p \in \mathcal{S}(\mathfrak{g})$ ,  $\beta(p) = \exp_* p$ . (Note that there is no Jacobian involved here.)

## 1.5. POISSON STRUCTURE

As is well known, the dual  $\mathfrak{g}^*$  of  $\mathfrak{g}$  is a Poisson manifold. For  $f, g$  functions on  $\mathfrak{g}^*$ , the Poisson bracket is

$$\{f, g\}(v) = \frac{1}{2}v([\mathrm{d}f(v), \mathrm{d}g(v)]) \quad (21)$$

for  $v = v_k e_k^* \in \mathfrak{g}^*$ , where  $\mathrm{d}f(v), \mathrm{d}g(v) \in \mathfrak{g}^{**}$  are identified with elements of  $\mathfrak{g}$ . Consider the structure constants relative to the given basis  $e_i$  of  $\mathfrak{g}$ :

$$[e_i, e_j] = c_{ij}^k e_k \quad (22)$$

(we use the Einstein convention of summing repeated indices). Writing  $\partial_j$  for the partial derivatives with respect to the dual basis  $e_j^*$  of  $e_i$ , we get the following formula for the Poisson bracket

$$\gamma(f, g)(v) = \{f, g\}(v) = \frac{1}{2}c_{ij}^k v_k \partial_i(f)(v) \partial_j(g)(v); \quad (23)$$

said otherwise, the corresponding Poisson tensor is linear:

$$\gamma^{ij}(v) = \frac{1}{2}c_{ij}^k v_k.$$

## 2. General Facts about the Kontsevich Construction of a Star Product

### 2.1. ADMISSIBLE GRAPHS

Consider, for every integers  $n \geq 1$  and  $m \geq 1$  a set  $G_{n,m}$  of labelled oriented graphs  $\Gamma$  with  $n+m$  vertices:  $1, \dots, n$  are the vertices of ‘first kind’,  $\bar{1}, \dots, \bar{m}$  are the vertices of ‘second kind’. Here labelled means that the edges are labelled. We will actually use these graphs here mostly for  $m=1$  (resp. 2). In those cases, the vertices of second kind will be named  $M$  (resp.  $L, R$ ). The set of vertices of  $\Gamma$  is denoted  $V_\Gamma$ , the set of vertices of the first kind  $V_\Gamma^1$ , and the set of edges  $E_\Gamma$ . For  $e = (a, b) \in E_\Gamma$ , we write  $a = a(e)$  and  $b = b(e)$ . For graphs in  $G_{n,m}$ , we assume that for any edge  $e$ ,  $a(e)$  is a vertex of the first kind,  $b(e) \neq a(e)$  and for every vertex  $a \in \{1, \dots, n\}$ , the set of edges starting at  $a$  has two elements, written  $e_a^1, e_a^2$ . Finally, we assume that there are no double edges.

As usual, a *root* of an oriented graph is a vertex which is the end-point of no edge, and a *leaf* is a vertex which is the beginning-point of no edge.

### 2.2. MULTIDIFFERENTIAL OPERATORS ASSOCIATED TO GRAPHS

Let  $X$  be a real vector space of dimension  $d$  with a chosen basis  $v_1, \dots, v_d$ . The vector field  $\partial_{v_j}$  on  $X$  is denoted  $\partial_j$ . We fix a  $C^\infty$  bi-vector field  $\alpha = \sum_{i,j \in \{1, \dots, d\}} \alpha^{ij} \partial_i \partial_j$  on  $X$ .



### 2.2.1. Differential Operators

Let  $\Gamma \in G_{n,1}$ . We define a differential operator  $D_{\Gamma,\alpha}$  by the formula:

$$D_{\Gamma,\alpha}(\phi) = \sum_{I \in \mathcal{L}} \left[ \prod_{k=1}^n \left( \prod_{\substack{e \in E_{\Gamma} \\ b(e)=k}} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right] \left( \prod_{\substack{e \in E_{\Gamma} \\ b(e)=M}} \partial_{I(e)} \right) \phi, \quad (24)$$

where  $\mathcal{L}$  is the set of maps from  $E_{\Gamma}$  to  $\{1, \dots, d\}$  (tagging of edges). One should think of this operator in the following way: for each tagging of edges, ‘put’ at each vertex of the first kind  $\ell$  the coefficient  $\alpha^{ij}$  corresponding to the tags of the edges originating at  $\ell$  and ‘put’  $\phi$  at  $M$ . Whatever is at a given vertex ( $\ell \in \{1, \dots, n\}$  or  $M$ ) should be differentiated according to the tags of edges ending at that vertex, and multiply over all vertices.

### 2.2.2. Bi and Multi-Differential Operators

Consider a graph  $\Gamma \in G_{n,2}$ . Define a bi-differential operator  $B_{\Gamma,\alpha}$  by the formula

$$B_{\Gamma,\alpha}(f, g) = \sum_{I \in \mathcal{L}} \left[ \prod_{k=1}^n \left( \prod_{\substack{e \in E_{\Gamma} \\ b(e)=k}} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right] \left( \prod_{\substack{e \in E_{\Gamma} \\ b(e)=L}} \partial_{I(e)} \right) f \left( \prod_{\substack{e \in E_{\Gamma} \\ b(e)=R}} \partial_{I(e)} \right) g. \quad (25)$$

The bi-differential operator  $B_{\Gamma,\alpha}$  has a similar interpretation as  $D_{\Gamma,\alpha}$ .

For graphs  $\Gamma$  in  $G_{n,m}$ , by a straightforward generalization of (25), we define  $m$ -differential operators, with the same interpretation as before.

## 2.3. LIE ALGEBRA CASE

### 2.3.1. Relevant Graphs

Assume that the vector space  $X = \mathfrak{g}^*$ , and that the bi-vector field  $\alpha$  is the Poisson bracket  $\gamma$ , so that  $\alpha^{ij}(v) = \gamma^{ij}(v) = \frac{1}{2}c_{ij}^k v_k$ . Since we use only this bi-vector field, we shall drop  $\alpha$  from the notation. Because of the linearity of the bi-vector field associated with the Poisson bracket, for the graphs  $\Gamma \in G_{n,m}$  for which there exists a vertex of the first kind  $\ell$  with at least two edges ending at  $\ell$ ,  $B_{\Gamma} = 0$ . We will call *relevant* the graphs which have at most one edge ending at any vertex of the first kind; since we are dealing exclusively with the Lie algebra case, we will henceforth change our notation slightly, and use the notation  $G_{n,m}$  for the set of relevant graphs.

2.3.2. Action on Distributions

Let  $\Gamma \in G_{n,2}$ , and let us interpret ‘graphically’ the operator  $\mathcal{F}(B_\Gamma)$  acting on pairs of distributions  $u, v$  on  $\mathfrak{g}$ . As in (25), the action is defined as a sum over all tagging of edges of terms obtained in the following way: ‘put’  $u$  and  $v$  at vertices  $L$  and  $R$ , put at any vertex of the first kind the distribution  $\mathbf{d}_{[e_i, e_j]}$ , where  $i, j$  are the tags of the edges originating at that vertex; any edge tagged by  $\ell$  gives a multiplication by the function  $e_\ell^*$  of the distribution sitting at the end point of that edge. And finally, we take the convolution of the distributions at all vertices.

2.4. THE KONTSEVICH STAR PRODUCT

Kontsevich defines a certain compactification  $\overline{C}_{n,m}^+$  of the configuration space of  $n$  points  $z_1, \dots, z_n$  in the Poincaré upper half space, with  $z_i \neq z_j$  for  $i \neq j$ , and  $m$  points  $y_1 < y_2 < \dots < y_m$  in  $\mathbb{R}$ , up to the action of the  $az + b$ -group for  $a \in \mathbb{R}^{+*}$  and  $b \in \mathbb{R}$ . To each graph  $\Gamma$  is associated a weight  $w_\Gamma$  which is an integral of some differential form on  $\overline{C}_{n,m}^+$ .

Let  $X = \mathbb{R}^d$  with a Poisson structure  $\gamma$  considered as a bi-vector field. Let  $A = C^\infty(X)$  the corresponding Poisson algebra. A  $\star$ -product on  $A$  is an associative  $\mathbb{R}[[\hbar]]$ -bilinear product on  $A[[\hbar]]$  given by a formula of the type:

$$(f, g) \mapsto f \star_\hbar g = fg + \hbar B_1(f, g) + \frac{1}{2!} \hbar^2 B_2(f, g) + \dots + \frac{1}{n!} \hbar^n B_n(f, g) + \dots, \tag{26}$$

where  $B_j$  are bi-differential operators and such that

$$f \star_\hbar g - g \star_\hbar f = 2\hbar \gamma(f, g) \tag{27}$$

modulo terms in  $\hbar^2$ .

The Kontsevich star product is given by formulas

$$B_n(f, g) = \sum_{\Gamma \in G_{n,2}} w_\Gamma B_\Gamma(f, g). \tag{28}$$

Note that each  $B_n(\cdot, \cdot)$ , being a *finite* sum, is a bi-differential operator.

2.4.1. Setting  $\hbar = 1$

As is explained in [3], one can set  $\hbar = 1$ , or, for that matter,  $\hbar = t$  for any  $t \in \mathbb{R}$  in the Kontsevich formula in the Lie algebra case. Indeed, for fixed  $f, g \in \mathcal{S}(\mathfrak{g})$  of degrees  $p, q$  respectively, the terms  $B_{\Gamma, \gamma}(f, g) = 0$  for  $n > p + q$ , so that the sum (26) actually involves a finite number of terms. This operation is written  $\star$  rather than  $\star_1$ .

Replace the Lie algebra  $\mathfrak{g}$  by  $\mathfrak{g}_t$  for  $t \in \mathbb{R}$ , or  $t$  a formal variable as in (6). It is easy to check that the Kontsevich  $\star$ -product  $\star_1$  for the Lie algebra  $\mathfrak{g}_t$  at  $\hbar = 1$  coincides with  $\star_t$ .

### 3. An Extension of the Kontsevich Star Product

#### 3.1. THE SYMBOL OF THE STAR PRODUCT

We consider here the star product as a formal bi-differential operator, i.e. as an element of  $\mathfrak{W}^2(\mathfrak{g}^*)[[\hbar]]$ . It has a symbol  $A_{\hbar}$  which belongs to  $\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g})[[\hbar]]$ . We will view  $A_{\hbar} = A_{\hbar}(X, Y)$  as a polynomial map of  $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$  into  $\mathcal{S}(\mathfrak{g})[[\hbar]]$  and prove some properties of  $A_{\hbar}$ .

Recall that, for  $\Gamma \in G_{n,2}$ ,  $\sigma_{\Gamma}$  is the symbol of the corresponding bi-differential operator  $B_{\Gamma}$ . Let

$$\sigma_n = \sum_{\Gamma \in G_{n,2}} w_{\Gamma} \sigma_{\Gamma}. \tag{29}$$

We have, for  $X, Y \in \mathfrak{g}$

$$A_{\hbar}(X, Y) = 1 + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sigma_n(X, Y). \tag{30}$$

We will describe a factorization of  $A_{\hbar}(X, Y)$  in Proposition 3.7 below. We first need to have a careful inspection of various graphs and their symbols.

##### 3.1.1. Wheels

We say that a graph  $\Gamma \in G_{n,m}$  contains a *wheel* of length  $p$  if there is a finite sequence  $\ell_1, \dots, \ell_p \in \{1, \dots, n\}$  with  $p \geq 2$  such that  $(\ell_1, \ell_2), \dots, (\ell_{p-1}, \ell_p), (\ell_p, \ell_1)$  are edges, and there are no other edges beginning at one of the  $\ell_k$  and ending at another  $\ell_{k'}$ . The graph  $\Gamma$  is a wheel if  $p = n$ .

##### 3.1.2. Simple Components

Let  $\Gamma \in G_{n,m}$ . Consider the graph  $\tilde{\Gamma}$  whose vertices are  $\{1, \dots, n\}$  and whose edges are those edges  $e \in E_{\Gamma}$  such that  $b(e) \notin \bar{1}, \dots, \bar{m}$ . Let  $\tilde{\Gamma}_i$  ( $i \in I$ ) be the connected components of  $\tilde{\Gamma}$ . The *simple* component  $\Gamma_i$  is the graph whose vertices are the vertices of  $\tilde{\Gamma}_i$  and  $\bar{1}, \dots, \bar{m}$ , and whose edges are the edges of  $\tilde{\Gamma}_i$  and the edges of  $\Gamma$  beginning at a vertex of  $\tilde{\Gamma}_i$ . It is easy to see that any simple component of a graph in  $G_{n,m}$  can be identified to a graph in  $G_{n',m}$  for  $n' \leq n$ .

In this situation, we use the notation

$$\Gamma = \coprod_i \Gamma_i, \tag{31}$$

and more generally, if  $\Gamma'$  and  $\Gamma''$  are sub-graphs whose simple components determine a partition of the  $\Gamma_i$ , we write

$$\Gamma = \Gamma' \sqcup \Gamma''. \tag{32}$$

A *simple graph* is a graph with only one simple component.

3.1.3. *Symbols*

We now give rules to compute the symbol of the operator attached to a graph  $\Gamma \in G_{n,m}$ . They are recorded as a series of lemmas. First, one observes:

LEMMA 3.1. *Let  $\Gamma \in G_{n,2}$  with  $r$  roots. The symbol  $\sigma_\Gamma$  is of total degree  $n+2r$ , of partial degree  $r$  for the  $\mathcal{S}(\mathfrak{g})$  components (we call this degree the polynomial degree) and of partial degree  $n+r$  for the  $\mathcal{S}(\mathfrak{g}^*) \otimes \mathcal{S}(\mathfrak{g}^*)$  component (the differential degree).*

The next two lemmas are essentially in [14]:

LEMMA 3.2. *If  $\Gamma \in G_{n,n}$  is a wheel of length  $n$ , then*

$$\sigma_\Gamma(X_1, \dots, X_n) = \text{tr}(\text{ad } X_1 \dots \text{ad } X_n). \quad (33)$$

LEMMA 3.3. *The symbol associated to the graph  $\Gamma \in G_{1,2}$  whose edges are  $(1, L)$  and  $(1, R)$  is the map*

$$(X, Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto \frac{1}{2}[X, Y].$$

Finally, the following statements can be easily derived from the definitions.

LEMMA 3.4. *Let  $\Gamma \in G_{n,m}$ . Assume that there exists a sub-graph  $\Gamma_0$  with  $p$  leaves  $\ell_1, \dots, \ell_p$  such that*

- *each leaf  $\ell_j$  is the end-point of one edge*
- *$\Gamma$  is the union of  $\Gamma_0$  and of  $p$  sub-graphs  $\Gamma_1, \dots, \Gamma_p$*
- *each  $\Gamma_j$  has a single root  $\ell_j$ .*

*Then*

$$\sigma_\Gamma = \sigma_{\Gamma_0}(\sigma_{\Gamma_1}, \dots, \sigma_{\Gamma_p}). \quad (34)$$

LEMMA 3.5. *Let  $\Gamma \in G_{n,m}$ . There exist two subgraphs  $\Gamma_1$  and  $\Gamma_2$  (possibly empty), with  $\Gamma_1$  having wheels and no roots, and  $\Gamma_2$  having roots and no wheels such that  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ . The decomposition is unique up to labelling of vertices.*

LEMMA 3.6. *Let  $\Gamma \in G_{n,m}$  and assume that  $\Gamma = \Gamma' \sqcup \Gamma''$ , then  $\sigma_\Gamma = \sigma_{\Gamma'} \sigma_{\Gamma''}$ .*

3.1.4. *Decomposition of  $A_{\hbar}$* 

Let us consider the following two subsets of  $G_{n,m}$ :

$$\begin{aligned} G_{n,2}^w &= \{\Gamma \in G_{n,2}, \Gamma \text{ with no roots}\}, \\ G_{n,2}^r &= \{\Gamma \in G_{n,2}, \Gamma \text{ with no wheels}\}, \end{aligned}$$

and consider the two following symbols:

$$A_{\hbar}^w(X, Y) = 1 + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sum_{\Gamma \in G_{n,2}^w} w_{\Gamma} \sigma_{\Gamma}(X, Y),$$

$$A_{\hbar}^r(X, Y) = 1 + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sum_{\Gamma \in G_{n,2}^r} w_{\Gamma} \sigma_{\Gamma}(X, Y).$$

**PROPOSITION 3.7.** (1) *As an  $\mathcal{S}(\mathfrak{g})$  valued map on  $\mathfrak{g} \times \mathfrak{g}$ ,  $A_{\hbar}^w$  is scalar valued.*  
 (2) *The symbol  $A_{\hbar}(X, Y)$  decomposes as a product*

$$A_{\hbar}(X, Y) = A_{\hbar}^w(X, Y)A_{\hbar}^r(X, Y).$$

*Proof.* (1) By Lemma 3.1, the symbol of a graph with no roots has polynomial degree 0. This proves the first assertion.

(2) By Lemma 3.5, any graph  $\Gamma$  is  $\Gamma_1 \sqcup \Gamma_2$ ,  $\Gamma_1$  with no roots,  $\Gamma_2$  with no wheels. It is easy to see from their definition that the weights are multiplicative [14, 6.4.1]:  $w_{\Gamma} = w_{\Gamma_1} w_{\Gamma_2}$ . Finally, the  $n!$  factors come from the labelling in the decomposition 3.5.  $\square$

For  $X, Y \in \mathfrak{g}_{\hbar}$ , let  $Z_{\hbar}(X, Y)$  be their Campbell–Hausdorff series. Writing  $Z = Z_1$ , we have

$$Z_{\hbar}(X, Y) = \hbar^{-1} Z(\hbar X, \hbar Y). \tag{35}$$

It is well known that the Campbell–Hausdorff series  $Z(X, Y)$  converges for  $X, Y$  small enough. Therefore, for any  $h_0$ , there exists  $\epsilon$  such that for  $\|X\|, \|Y\| \leq \epsilon$ , the power series (in  $\hbar$ )  $Z_{\hbar}$  converges in norm for  $|\hbar| \leq |h_0|$ .

The following is due to V. Kathotia [13, Theorem 5.0.2]:

**PROPOSITION 3.8.**  $A_{\hbar}^r(X, Y) = e^{Z_{\hbar}(X, Y) - X - Y}$ .

As a consequence, for  $\|X\|$  and  $\|Y\|$  small enough, the formal series  $A_{\hbar}^r$  converges for  $\hbar = 1$ .

**PROPOSITION 3.9.** *Considered as an  $\mathcal{S}(\mathfrak{g})$ -valued function on  $\mathfrak{g} \times \mathfrak{g}$ ,  $A_{\hbar}^w(X, Y)$  is a convergent series in a neighborhood of  $(0, 0)$ . Moreover,  $A_{\hbar}^w(X, Y) = \exp(A_{\hbar}^{(w)}(X, Y))$  where  $A_{\hbar}^{(w)}$  is the contribution to the series corresponding to graphs with exactly one wheel.*

*Proof.* It's easy to see that graphs with exactly one wheel (and no roots) correspond to simple graphs of wheel type. For such  $\Gamma$  with  $n_{\Gamma}$  vertices of first type, denote by  $m_{\Gamma}$  the cardinal of his group of symmetries (a subgroup of the group

of permutations of  $n_\Gamma$  elements). Let us call geometric a graph without labels and numberings. A geometric graph without roots has a decomposition

$$\underbrace{\Gamma_1 \amalg \cdots \amalg \Gamma_1}_{k_1} \amalg \underbrace{\Gamma_2 \amalg \cdots \amalg \Gamma_2}_{k_2} \cdots \amalg \underbrace{\Gamma_p \amalg \cdots \amalg \Gamma_p}_{k_p}$$

and appears

$$\frac{(k_1 n_{\Gamma_1} + \cdots + k_p n_{\Gamma_p})!}{k_1! (m_{\Gamma_1})^{k_1} \cdots k_p! (m_{\Gamma_p})^{k_p}}$$

times in  $A_{\hbar}^w(X, Y)$  due to the various numberings of vertices and  $2^{k_1 n_1 + \cdots + k_p n_p}$  due to the numberings of arrows. So one gets\*

$$\begin{aligned} A_{\hbar}^w(X, Y) &= 1 + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sum_{\Gamma \in G_{n,2}^w} w_\Gamma \sigma_\Gamma(X, Y) \\ &= 1 + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sum_{\substack{k_i, n_i, \sum_i k_i n_i = n \\ \Gamma_i \text{ simple geometric} \\ \text{of wheel type } (n_i, 2)}} n! \prod_i w_{\Gamma_i}^{k_i} \frac{(2^{n_i} \sigma_{\Gamma_i}(X, Y))^{k_i}}{k_i! m_{\Gamma_i}^{k_i}} \\ &= \exp \left( \sum_{n \geq 1} \hbar^n \sum_{\Gamma \text{ simple geometric} \\ \text{of wheel type } (n, 2)} w_\Gamma \frac{2^n \sigma_\Gamma(X, Y)}{m_\Gamma} \right) \\ &= \exp(A_{\hbar}^{(w)}(X, Y)). \end{aligned} \tag{36}$$

We now need to prove convergence of the series  $A_{\hbar}^{(w)}(X, Y)$ , and by homogeneity we can set  $\hbar = 1$ . Without loss of generality, we can assume that the structure constants  $|c_{ij}^k| \leq 2$ . Therefore,

$$\frac{1}{2^p} |\text{tr}(\text{ad } e_{i_1} \text{ ad } e_{i_2} \cdots \text{ad } e_{i_p})| \leq d^p$$

(where  $d = \dim \mathfrak{g}$ ) and, more generally,

$$\frac{1}{2^p} |\text{tr}(\text{ad } z_1 \text{ ad } z_2 \cdots \text{ad } z_k)| \leq d^p,$$

with  $z_j$  Lie monomial in  $e_i$  of degree  $p_j$  and  $\sum p_j = p$ .

Let  $\Gamma \in G_{n,2}^w$  with only one wheel of length  $p \leq n$ . If the absolute values of all components  $x_i, y_j$  of  $X, Y$  respectively are less than  $r$ ,

$$|\sigma_\Gamma(X, Y)| \leq r^n d^n d^n = r^n d^{2n}. \tag{37}$$

\*In [27,28] one should correct the definition of density function with geometric graphs by this factor  $1/m_\Gamma$ .

Besides, the following inequalities can be found in [3, Lemma 2.2 and 2.3]:

$$|w_\Gamma| \leq 4^n, \quad |G_{n,2}| \leq (8e)^n n!.$$

Finally, the terms of the series in  $A_1^{(w)}(X, Y)$  can be bounded by  $(32e)^n r^n d^{2n}$ , which proves convergence for  $r$  small enough.  $\square$

### 3.2. A FORMULA FOR THE STAR PRODUCT

**PROPOSITION 3.10.** *Let  $t \in \mathbb{R}$ . Let  $u, v \in \mathcal{S}(\mathfrak{g})$ , and  $u \star_t v$  their star product, considered as distributions on  $\mathfrak{g}$ . Then  $u \star_t v$  is given by the*

$$\langle u \star_t v, \phi \rangle = \langle u \otimes v, A_t^w \cdot (\phi \circ Z_t) \rangle \quad (38)$$

for  $\phi$  a test function on  $\mathfrak{g}$ .

In this proposition, we are of course looking at the Fourier transform of  $\star_t$  (see 20), meaning that we are actually expressing  $\mathbf{d}_{u \star_t v}$  in terms of  $\mathbf{d}_u, \mathbf{d}_v$ , but we avoid using these cumbersome notations. (See also [5] for a similar ‘integral’ formula.)

*Proof.* Since for fixed  $u, v$  the series for  $u \star_{\hbar} v$  is finite, we can substitute to  $\hbar$  a real number  $t$ , and interchange summation with the test function-distribution bracket:

$$\langle u \star_t v, \phi \rangle = \langle u(X) \otimes v(Y), [\partial_{A_t(X, Y)} \phi](X + Y) \rangle, \quad (39)$$

and we avoid convergence problems for a fixed  $t$  by taking a test function  $\phi$  with small enough support. We know that  $A_t(X, Y) = A_t^w(X, Y)A_t^r(X, Y)$ , and  $A_t^w(X, Y)$  is scalar valued (Proposition 3.7). So

$$[\partial_{A_t(X, Y)} \phi](X + Y) = A_t^w(X, Y)[\partial_{A_t^r(X, Y)} \phi](X + Y)$$

By Proposition 3.8,  $A_t^r(X, Y) = e^{Z_t(X, Y) - X - Y}$ , and by Taylor’s formula for polynomials,

$$\partial_{A_t^r(X, Y)} \phi(X + Y) = \phi(X + Y + Z_t(X, Y) - X - Y) = \phi(Z_t(X, Y)). \quad (40)$$

The proposition follows.  $\square$

### 3.3. AN EXTENSION OF THE KONTSEVICH STAR PRODUCT

In this section, we use formula (38) to *define* the star product of distributions with small enough compact support. By Proposition 3.10, the definition agrees with the original one for distributions with point support.

**PROPOSITION AND DEFINITION 3.11.** *Let  $t$  be a fixed real number. Then for all distributions  $u$  and  $v$  on  $\mathfrak{g}$  with sufficiently small support, the formula (38):*

$$\phi \mapsto \langle u \star_t v, \phi \rangle = \langle u \otimes v, A_t^w(X, Y) \cdot \phi(Z_t(X, Y)) \rangle$$

*defines a distribution  $u \star_t v$  on  $\mathfrak{g}$ .*

*Proof.* Let  $K_u$  and  $K_v$  be the supports of  $u$  and  $v$ . Assume that  $K_u$  and  $K_v$  are included in a ball or radius  $\alpha$ . Assume that  $2\alpha$  is less than the radius of convergence of  $A_t(X, Y)$ . It is clear that (38) makes sense. We need to prove that the functional  $t$  defined therein is a distribution. Since  $(X, Y) \mapsto Z_t(X, Y)$  is analytic from  $\mathfrak{g} \times \mathfrak{g}$  to  $\mathfrak{g}$ , the pushforward of the compactly supported distribution  $u \otimes v$  under  $Z_t$  is a distribution. But  $u \star_t v$  is obtained by multiplying this distribution by the analytic function  $A_t^w$ , so it is a distribution.  $\square$

This proves the first statement of Theorem 0.2.

### 3.4. A CONNECTION BETWEEN THE STAR PRODUCT AND GROUP CONVOLUTION

An element of  $\mathcal{S}((\mathfrak{g}^*)) = \mathbb{R}[[\mathfrak{g}]]$  play a crucial role in this situation:

$$q(X) = \left( \det_{\mathfrak{g}} \frac{e^{\text{ad } X/2} - e^{-\text{ad } X/2}}{\text{ad } X} \right)^{1/2}. \tag{41}$$

It is clear by definition that  $q(X)$  is analytic on  $\mathfrak{g}$ . Consider the ‘infinite order constant coefficient differential operator’  $\partial_q$  with symbol  $q$  (written  $I_{\text{st}}$  in [14]).

In [14] another element of  $\mathcal{S}((\mathfrak{g}^*))$  plays a role:

$$\tau(X) = \exp \left( \sum_n \frac{w_n}{n} \text{tr}((\text{ad } X)^n) \right),$$

where  $w_n$  is the weight corresponding to the graph with one pure wheel of order  $n$ . Since by [24]  $\tau(X) = 1$  for all  $X$ , we will not mention it further.

As in [14], we write  $I_{\text{alg}}$  for the isomorphism from  $(\mathcal{U}(\mathfrak{g}), *_G)$  to  $(\mathcal{S}(\mathfrak{g}), \star)$  coming from (27) and the universal property of  $\mathcal{U}(\mathfrak{g})$ . Recall that  $\beta$  is the symmetrization map ( $I_{\text{PBW}}$  in [14]). By [14, 24] these three maps are related by the following:

$$I_{\text{alg}}^{-1} = \beta \circ \partial_q. \tag{42}$$

When we consider elements in  $\mathcal{S}(\mathfrak{g})$  and  $\mathcal{U}(\mathfrak{g})$  as distributions,  $\partial_q$  should be replaced by multiplication operators, so that (42) is equivalent to

$$I_{\text{alg}}^{-1}(p) = \beta(pq), \tag{43}$$

for  $p \in \mathcal{S}(\mathfrak{g})$ , and  $\beta$  is interpreted as the direct image of distributions under the exponential map. Using (2), and replacing everywhere  $\mathfrak{g}$  by  $\mathfrak{g}_t$ , we therefore have the identity (13):

$$\eta_t^{-1}(\eta_t(u) *_G \eta_t(v)) = u \star_t v,$$

for  $u, v$  distributions on  $\mathfrak{g}$  supported at 0.



The following Proposition finishes the proof of Theorem 0.2.

**PROPOSITION 3.12.** (1) *The scalar valued function  $A^w$  is given by*

$$A^w(X, Y) = \frac{q(X)q(Y)}{q(Z(X, Y))}. \quad (44)$$

(2) *Let  $u, v$  be distributions on  $\mathfrak{g}$  with (small enough) compact support. The identity (13):*

$$\eta_t^{-1}(\eta_t(u) *_{G_t} \eta_t(v)) = u \star_t v$$

holds.

*Proof.* (1) Let  $u, v \in \mathcal{S}(\mathfrak{g})$ . The convolution  $\beta(uq) *_{G} \beta(vq)$  is given by

$$\langle \beta(uq) *_{G} \beta(vq), \psi \rangle = \langle uq \otimes vq, \psi \circ \exp Z \rangle, \quad (45)$$

for  $\psi$  a test function on  $G$ . Since  $I_{\text{alg}}$  is an algebra homomorphism, by (42) we get

$$\beta((u \star v)q) = \beta(uq) *_{G} \beta(vq). \quad (46)$$

Therefore, applying (38), we get for any distributions  $u, v$  with support 0:

$$\langle u \otimes v, A^w \cdot (q \circ Z) \psi \circ \exp Z \rangle = \langle uq \otimes vq, \psi \circ \exp Z \rangle. \quad (47)$$

Fix a neighborhood of  $(0, 0)$  in  $\mathfrak{g} \times \mathfrak{g}$  on which  $A_w(X, Y)$  is defined. Assume that  $\psi(\exp(Z(X, Y))) \equiv 1$  on that neighborhood. We have

$$\langle u \otimes v, A^w \cdot (q \circ Z) \rangle = \langle u \otimes v, q \otimes q \rangle. \quad (48)$$

Since this equality holds for any  $u$  and  $v$  supported at 0, this implies that the two functions  $A^w(X, Y)q(Z(X, Y))$  and  $q(X)q(Y)$  have same derivatives at  $(0, 0)$ ; since they are analytic, they must be equal.

(2) The second statement follows immediately from (45), (44) and (38).  $\square$

We will prove now one last property about the extended star product, relating it directly to graphs:

**PROPOSITION 3.13.** *Let  $u, v$  be two distributions with small enough compact support on  $\mathfrak{g}$ . The following holds for any integer  $n \geq 1$ :*

$$\left. \frac{d^n}{dt^n} \right|_{t=0} u \star_t v = \sum_{\Gamma \in G_{n,2}} (u, v) \cdot \mathcal{F}(w_{\Gamma} B_{\Gamma}). \quad (49)$$

Before giving the proof, let us observe that we fall short from proving that  $u \star_t v$  is *analytic* in  $t$  for  $u, v$  general. However, we will prove such a result for  $u, v$  invariant in Section 4.

*Proof.* For  $u, v$  be two distributions on  $\mathfrak{g}$  with small support, and  $\phi$  a test function on  $\mathfrak{g}$ , let us define

$$\langle C_n(u, v), \phi \rangle = \left. \frac{d^n}{dt^n} \right|_{t=0} \langle u \star_t v, \phi \rangle. \quad (50)$$

By inspection,  $C_n$  is a bi-differential operator with polynomial coefficient acting on distributions.

For distributions  $u, v \in \mathcal{S}(\mathfrak{g})$ , we *know* that

$$C_n(u, v) = \sum_{\Gamma \in G_{n,2}} (u, v) \cdot \mathcal{F}(w_\Gamma B_\Gamma). \quad (51)$$

Now bi-differential operators with polynomial coefficients are completely determined by their action on distributions with point support. This proves that

$$C_n = \sum_{\Gamma \in G_{n,2}} \mathcal{F}(w_\Gamma B_\Gamma). \quad (52)$$

□

## 4. Proof of Theorem 0.3

### 4.1. CONVOLUTION ON THE LEVEL OF GERMS

The first step is to transfer the results of Section 3 to germs. We will begin by giving a few definitions.

#### 4.1.1. Germs

Recall that the *germ at 0* (resp. at 1) of a distribution  $u$  on  $\mathfrak{g}$  (resp.  $G$ ) is the equivalence class of  $u$  for the equivalence relation  $u_1 \sim u_2$  if and only if there exists a neighborhood  $C$  of 0 (resp. 1) such that for any test function  $\phi$  on  $\mathfrak{g}$  (resp.  $G$ ) with support in  $C$ ,  $\langle u_1, \phi \rangle = \langle u_2, \phi \rangle$ . Clearly, for any distribution  $u$  and any given neighborhood  $\Omega$  of 0 (resp. 1), there exists a distribution with support in  $\Omega$  defining the same germ at 0 as  $u$ .

#### 4.1.2. Action of $G$ on Germs

From the action of  $G$  on  $G$  by conjugation (resp. the adjoint action on  $\mathfrak{g}$ ) we get an action of  $G$  on functions and distributions on  $G$  (resp.  $\mathfrak{g}$ ). For  $g \in G$  and  $u$  a distribution, we write  $u^g$  for the image of  $u$  under  $g$ . We get therefore an infinitesimal action of  $\mathfrak{g}$  on functions and distributions on  $\mathfrak{g}$ , which is exactly the action by adjoint vector fields  $\text{adj}_X$  for  $X \in \mathfrak{g}$ . It is straightforward to see that these actions descend to germs.

4.1.3. *Invariant Germs*

Invariant germs are defined as germs  $u$  such that  $u \cdot A = 0$  for all  $A \in \mathfrak{g}$ . By taking a basis of  $\mathfrak{g}$ , it is easy to see

LEMMA 4.1. *A germ  $u$  is invariant if and only if, for any distribution  $\phi$  representing  $u$ , there exists an open neighborhood  $\Omega$  of 0 such that, for any  $\phi$  supported in  $\Omega$  and any  $A \in \mathfrak{g}$*

$$\langle u \cdot A, \phi \rangle = 0. \tag{53}$$

4.1.4. *The Compatibility Condition*

For  $u$  a germ on  $\mathfrak{g}$ , recall we have defined  $C_0[u]$  as the cone  $C_0(\text{supp } u)$  for any  $u$  representing  $u$ . Two germs  $u, v$  are compatible if

$$C_0[u] \cap -C_0[v] = \{0\}. \tag{54}$$

4.1.5. *Proof of the First Statement of Theorem 0.3*

We need to prove that the star product descends to germs. The analogous statements about the convolutions on  $\mathfrak{g}$  and on  $G$  are made in [12]. By formula (13), we deduce it for  $u \star v$ .

4.2. INVARIANT GERMS

In order to finish the proof of Theorem 0.3, we need to have very precise statements about the choice of representatives of invariant germs that we will work with. This is what we do in this section.

We choose some norm  $\| \cdot \|$  on  $\mathfrak{g}$ , and write  $B(0, r)$  for the open ball of center 0 and radius  $r$ . For us, *open cone* means cone containing 0 with open intersection with  $B(0, 1) \setminus \{0\}$ .

LEMMA 4.2. (1) *Let  $u$  be a germ. For any open cone  $D$  containing  $C_0[u]$ , there exists a representative of  $u$  with support included in  $D$ .*

(2) *Let  $u, v$  be compatible germs. There exist open cones  $D_0[u] \supset C_0[u]$ ,  $D_0[v] \supset C_0[v]$  such that*

$$\overline{D_0[u]} \cap -\overline{D_0[v]} = \{0\}. \tag{55}$$

*Proof.* The second statement is easy. We prove the first. Let  $u_1$  be any representative of  $u$ . By definition of  $C_0[u]$ , there exists  $\eta > 0$  such that  $\text{supp } u_1 \cap B(0, \eta) \subset D$ . Let  $\chi$  be a  $C^\infty$  function supported in  $B(0, \eta)$  which is identically equal to 1 in  $B(0, \eta/2)$ . Clearly,  $u = u_1 \chi$  is a representative of  $u$  with support included in  $D$ .  $\square$

For  $\beta > 0$ , we write  $D_0^\beta[u] = D_0[u] \cap B(0, \beta)$ . The following lemma is crucial for our purposes.

LEMMA 4.3. *Let  $u, v$  be compatible germs, and  $D_0[u], D_0[v]$  open cones as in Lemma 4.2. There exists a  $\beta > 0$  such that, for any  $\gamma > 0$ , there exists  $\delta > 0$  satisfying*

$$\text{supp}(\phi \circ Z_t) \cap (D_0^\beta[u] \times D_0^\beta[v]) \subset B(0, \gamma) \times B(0, \gamma), \tag{56}$$

for all smooth  $\phi$  with support in  $B(0, \delta)$ .

*Proof.* We choose open cones  $D_1[u], D_1[v]$  containing  $\overline{D_0[u]}, \overline{D_0[v]}$  respectively, and verifying

$$\overline{D_1[u]} \cap \overline{D_1[v]} = \{0\}.$$

We study the restriction of  $Z_t(X, Y)$  to  $D_1[u] \times D_1[v]$ . The Campbell–Hausdorff formula implies that there exists a positive number  $\beta_1$  such that the  $\mathfrak{g}$ -valued map  $(t, X, Y) \mapsto Z_t(X, Y)$  is analytic for  $t \leq 1, \|X\| \leq \beta_1, \|Y\| \leq \beta_1$ . Furthermore,

$$\frac{\partial Z_t}{\partial X}(0, 0) = \frac{\partial Z_t}{\partial Y}(0, 0) = I. \tag{57}$$

Therefore, the implicit equation  $Z_t(X, Y) = 0$  can be solved: there exists a constant  $\beta_2 < \beta_1$  and an analytic map  $(X, t) \mapsto z_t(X)$ , defined for  $t \in [-1, 1]$  and  $\|X\| \leq \beta_2$  such that  $Z_t(X, Y) = 0$  is equivalent to  $Y = z_t(X)$  for  $\|X\| \leq \beta_2, \|Y\| \leq \beta_2$ . Now since  $\partial z_t / \partial X(0) = -I$  and  $D_1[u]$  is an open cone, we conclude that there exists  $\beta < \beta_2$  such that, for  $X \in D_0^\beta[u]$ , and any  $t \in [-1, 1]$ ,  $z_t(X) \in -D_1[u]$ . Using (55), for  $X \in D_0^\beta[u], Y \in D_0^\beta[v], Z_t(X, Y) = 0$  implies  $X = Y = 0$ . Using a compactness argument, we deduce

$$\begin{aligned} \exists \beta, \forall \gamma \in (0, \beta], \exists \delta > 0, \quad \forall X \in D_0^\beta[u], \forall Y \in D_0^\beta[v], \\ \forall t \in [-1, 1], \|Z_t(X, Y)\| \leq \delta \text{ implies } \|X\| \leq \gamma, \|Y\| \leq \gamma. \end{aligned} \tag{58}$$

Assume that  $\phi$  is a test function supported in  $B(0, \delta)$ . Then, by (58),

$$\text{supp}(\phi \circ Z_t) \cap (D_0^\beta[u] \times D_0^\beta[v])$$

is included in  $B(0, \gamma) \times B(0, \gamma)$ , as we wanted. □

We can now make the following precise statement about the choice of representatives of germs.

PROPOSITION 4.4. *Let  $u, v$  be compatible germs, and  $D_0[u], D_0[v]$  open cones as in Lemma 4.2. There exists a positive real  $\beta$  such that, for any  $t \in [-1, 1]$ , the germ at 0 of  $u \star_t v$  does not depend on the choice of  $u$  (resp.  $v$ ) distribution supported in  $D_0^\beta[u]$  (resp.  $D_0^\beta[v]$ ) representing  $u$  (resp.  $v$ ). This implies that the germ of  $u \star v$  is independent of the choice of  $D_0[u], D_0[v]$ .*

*Proof.* Let  $u, v$  be compatible germs, and  $(u_1, v_1), (u_2, v_2)$  two pairs of representatives. By formula (38), since the multiplication factor  $A_t^w$  does not play a role, it is enough to prove, for any  $t \in [-1, 1]$ , the formula

$$\langle u_1 \otimes v_1, \phi \circ Z_t \rangle = \langle u_2 \otimes v_2, \phi \circ Z_t \rangle \quad (59)$$

provided the support of  $\phi$  is small enough. Assume that  $(u_1, u_2)$  (resp.  $(v_1, v_2)$ ) are supported in  $D_0^\beta[u]$  (resp.  $D_0^\beta[v]$ ). Since  $u_1 \sim u_2, v_1 \sim v_2$ , there exists  $\gamma$  such that for any test function  $\psi$  with support included in the ball  $B(0, \gamma)$ ,

$$\langle u_1, \psi \rangle = \langle u_2, \psi \rangle, \quad \langle v_1, \psi \rangle = \langle v_2, \psi \rangle. \quad (60)$$

From  $\gamma$ , we get  $\delta$  by Lemma 4.3. Using (56) we deduce (59) for  $\phi$  supported in  $B(0, \delta)$ .  $\square$

**PROPOSITION 4.5.** *Let  $u, v$  be two compatible invariant germs. The germ  $u \star_t v$  is invariant.*

*Proof.* It is enough to do it for  $t = 1$ . We consider  $\beta$  from Proposition 4.4. As before, we chose representatives  $u, v$  of  $u, v$  supported in  $D_0^\beta[u]$  and  $D_0^\beta[v]$  respectively. Since the germs are invariant, there exists a  $\gamma < \beta$  such that, for any test function  $\psi$  supported in  $B(0, \gamma)$

$$\langle u \cdot A, \psi \rangle = \langle v \cdot A, \psi \rangle = 0, \quad (61)$$

for all  $A \in \mathfrak{g}$ . Applying again Lemma 4.3, consider  $\phi$  a test function supported in  $B(0, \delta)$ . We need to prove that

$$\langle (u \star v) \cdot A, \phi \rangle = \langle (u \star v), A \cdot \phi \rangle = 0,$$

for all  $A \in \mathfrak{g}$ . Indeed, since the function  $q$  is invariant, it is enough to prove

$$\langle u \otimes v, (A \cdot \phi) \circ Z \rangle = 0. \quad (62)$$

Using the covariance of  $Z(X, Y)$  under the adjoint action of  $g \in G$ :

$$g.(Z(X, Y)) = Z(g.X, g.Y),$$

writing  $g = \exp(tA)$  and differentiating at  $t = 0$ , we get

$$(A \cdot \phi) \circ Z = (A \otimes 1 + 1 \otimes A)(\phi \circ Z). \quad (63)$$

Thus we get

$$\langle u \otimes v, (A \cdot \phi) \circ Z \rangle = \langle u \cdot A \otimes v, \phi \circ Z \rangle + \langle u \otimes v \cdot A, \phi \circ Z \rangle.$$

By Lemma 4.3

$$\text{supp}(\phi \circ Z) \cap (D_0^\beta[u] \times D_0^\beta[v]) \subset B(0, \gamma) \times B(0, \gamma).$$

Now we use (61) to conclude.  $\square$

## 4.3. END OF PROOF OF THEOREM 0.3

As before, we consider two compatible invariant germs  $u$  and  $v$ , and take representatives  $u, v$  of  $u, v$ . We shall prove the equivalence of distributions:

$$u \star_t v \sim u \star_{\mathfrak{g}} v, \quad (64)$$

for  $u$ , and  $v$  adequately chosen. It clearly will imply Theorem 0.3. It is enough to prove that for any arbitrary test function  $\phi$  with sufficiently small support

$$\langle u \star_t v, \phi \rangle = \langle u \star_{\mathfrak{g}} v, \phi \rangle. \quad (65)$$

Considering both sides as functions of  $t$ , we will prove that they are analytic in  $t$ , and that their derivatives at 0 at any order are equal.

## 4.3.1. Analyticity

We will derive the required analyticity result by extending some arguments of Rouvière from [22,23]. For the reader's convenience, we summarize below the results that we need from these papers.

Rouvière<sup>\*</sup> defines an analytic function  $e(X, Y)$ <sup>\*\*</sup> on a neighborhood of  $(0, 0)$  in  $\mathfrak{g} \times \mathfrak{g}$  and a family of maps  $\Phi_t$  (depending smoothly on  $t \in [0, 1]$ ) from a neighborhood of  $(0, 0)$  in  $\mathfrak{g} \times \mathfrak{g}$  to  $\mathfrak{g} \times \mathfrak{g}$  with the following properties

- $\Phi_0 = I$ .
- For all  $t$ ,  $\Phi_t(0, 0) = (0, 0)$  and  $\Phi_t$  is a local diffeomorphism at  $(0, 0)$ .
- If  $\sigma: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is the addition map, then  $\sigma \circ \Phi_t^{-1} = Z_t$ .

For a smooth function  $g$  on  $\mathfrak{g} \times \mathfrak{g}$ , define  $g_t = (f_t g) \circ \Phi_t^{-1}$ , where

$$f_t(X, Y) = \frac{q(tX)q(tY)}{q(tX+tY)} e(tX, tY)^{-1}.$$

Rouvière proves

$$\frac{\partial}{\partial t} g_t = \text{tr}_{\mathfrak{g}} \left( \text{ad } X \circ \frac{\partial}{\partial X} (g_t F_t) + \text{ad } Y \circ \frac{\partial}{\partial Y} (g_t G_t) \right), \quad (66)$$

where  $F_t$  and  $G_t$  are certain smooth functions on  $\mathfrak{g} \times \mathfrak{g}$ , and all differentials are taken at  $(X, Y)$ . Since the differential operators on the right can be expressed in terms of adjoint vector fields, Rouvière uses this to conclude that, for invariant distributions  $u$  and  $v$ ,

$$\left\langle u \otimes v, \frac{\partial}{\partial t} g_t \right\rangle \equiv 0, \quad \text{whence } \langle u \otimes v, g_0 \rangle = \langle u \otimes v, g_1 \rangle. \quad (67)$$

<sup>\*</sup>Rouvière's work is for symmetric spaces, but we apply it in the special case of Lie algebras.

<sup>\*\*</sup>The Kashiwara-Vergne combinatorial conjecture for Lie algebras is equivalent to the statement:  $e(X, Y) = 1$ .

If now  $\phi$  is a smooth function on  $\mathfrak{g}$ , applying this result to

$$g(X, Y) = \frac{e(X, Y)}{q(X)q(Y)}\phi(X + Y),$$

and pairing with  $uq$  and  $vq$ , we obtain

$$\langle u \otimes v, e(\phi \circ \sigma) \rangle = \langle uq \otimes vq, (\phi q^{-1}) \circ Z \rangle. \quad (68)$$

Rewriting this, we get

$$\langle u \otimes v, e(\phi \circ \sigma) \rangle = \langle \eta^{-1}(\eta(u) *_{\mathcal{G}} \eta(v)), \phi \rangle. \quad (69)$$

**LEMMA 4.6.** *Let  $u, v$  be two compatible invariant germs,  $D_0[u], D_0[v]$  chosen as in Lemma 4.2, and  $\beta$  from (56). Let  $u, v$  be representatives which verify the conditions of Proposition 4.4. There exists a positive number  $\delta$  such that, for any test function  $\phi$  supported in  $B(0, \delta)$ ,*

$$\langle u \star_t v, \phi \rangle \quad (70)$$

*is an analytic function of  $t$  in a neighborhood of  $[0, 1]$ .*

Note that this lemma implies that the function  $\Psi_{u,v,\phi}(t)$  considered in the introduction is analytic.

*Proof.* The first step is to prove that (69) holds under the assumptions of the lemma, provided  $\phi$  has small enough support. We proceed as in Proposition 4.5: from the invariance of  $u, v$ , we have a constant  $\gamma$  such that

$$\langle u \cdot A, \psi \rangle = \langle v \cdot A, \psi \rangle = 0, \quad (71)$$

for all  $A \in \mathfrak{g}$  and any  $\psi$  supported in  $B(0, \gamma)$ . We derive  $\delta$  from Lemma 4.3. Let  $\phi$  have support in  $B(0, \delta)$ . In Rouvière's proof which was just outlined above, the distribution  $u \otimes v$  is paired with the function  $g_t$  and some derivatives of  $g_t$ . Since  $\sigma \circ \Phi_t^{-1} = Z_t$ , we can write  $g_t = K_t(\phi \circ Z_t)$ , where  $K_t$  is some smooth function. Therefore, the support of  $g_t$  (and any derivative) is included in the support of  $\phi \circ Z_t$ . By (56) we see that

$$\text{supp } g_t \cap (D_0^\beta[u] \times D_0^\beta[v]) \quad (72)$$

is included in  $B(0, \gamma) \times B(0, \gamma)$ . Since we are pairing with  $u \otimes v$ , with  $u, v$  verifying (71), we still conclude as in [23, Theorem 2.1]) that  $\langle u \otimes v, g_t \rangle$  is independent of  $t$ , so that (67) and (69) still hold.

Now we apply (69), but to the algebra  $\mathfrak{g}_t$  (note that this  $t$  is not ‘the same’ as the  $t$  used before !). Using (13), since the  $e$  function for  $\mathfrak{g}_t$  is given by  $e_t(X, Y) = e(tX, tY)$  (see [22, Proposition 3.14]) we derive

$$\langle u \star_t v, \phi \rangle = \langle u(X) \otimes v(Y), e(tX, tY)\phi(X + Y) \rangle. \quad (73)$$

Using the fact that  $e$  and  $q$  are analytic in a neighborhood of 0, we conclude from (73) that  $\langle u \star_t v, \phi \rangle$  is an analytic function of  $t \in [-1, 1]$ .  $\square$

The analyticity of the right-hand side of (65) being immediate, it now remains to prove the equality of derivatives of both sides of (65) at 0 to all orders.

#### 4.3.2. Cancellation

Recall that from Proposition 3.13 that we have for  $u, v$  be two distributions with small enough compact support on  $\mathfrak{g}$  and any integer  $n \geq 1$ :

$$\left. \frac{d^n}{dt^n} \right|_{t=0} u \star_t v = (u, v)\mathcal{F}(B_n), \quad (74)$$

with  $B_n = \sum_{\Gamma \in G_{n,2}} w_\Gamma B_\Gamma$ .

We now define a certain subset  $\mathfrak{J}$  of  $\mathfrak{W}^2(\mathfrak{g}^*)$ . Considering the natural surjective map  $\mathcal{C}$  from  $\mathfrak{W}(\mathfrak{g}^*) \otimes \mathfrak{W}(\mathfrak{g}^*)$  to  $\mathfrak{W}^2(\mathfrak{g}^*)$  given by

$$p_1 \otimes q_1 \otimes p_2 \otimes q_2 \mapsto p_1 p_2 \otimes q_1 \otimes q_2$$

for  $p_i \in \mathcal{S}(\mathfrak{g})$ ,  $q_i \in \mathcal{S}(\mathfrak{g}^*)$ . Let  $B = \mathcal{C}(D_1 \otimes D_2)$ , then

$$\langle (u, v) \cdot \mathcal{F}(B), \phi \rangle = \langle u \cdot \mathcal{F}(D_1) \otimes v \cdot \mathcal{F}(D_2), \phi \circ \sigma \rangle. \quad (75)$$

Let now  $\mathfrak{J}$  be the left ideal in  $\mathfrak{W}(\mathfrak{g}^*)$  generated by adjoint vector fields; we define

$$\mathfrak{J} = \mathcal{C}(\mathfrak{J} \otimes \mathfrak{W}(\mathfrak{g}^*) + \mathfrak{W}(\mathfrak{g}^*) \otimes \mathfrak{J}). \quad (76)$$

We now establish the following Lemma:

LEMMA 4.7. *For any  $B \in \mathfrak{J}$ , and  $u, v, \phi$  as in Proposition 4.5,*

$$\langle (u, v)\mathcal{F}(B), \phi \rangle = 0. \quad (77)$$

*Proof.* By symmetry, it is enough to prove that for  $B = \mathcal{C}(D_1 A \otimes D_2)$  with  $A$  an adjoint vector field,  $D_1 \in \mathfrak{W}(\mathfrak{g}^*)$ ,  $D_2 \in \mathfrak{W}(\mathfrak{g}^*)$ ,

$$\langle (u, v) \cdot \mathcal{F}(B), \phi \rangle = 0. \quad (78)$$

But

$$\langle (u, v) \cdot \mathcal{F}(B), \phi \rangle = \langle u \cdot A \otimes v, \mathcal{C}(D_1 \otimes D_2)(\phi \circ \sigma) \rangle. \quad (79)$$

Since the support  $\mathcal{C}(D_1 \otimes D_2)(\phi \circ \sigma)$  is included in the support of  $\phi \circ \sigma$ , we can now apply Equation (60).  $\square$



To conclude the proof of Theorem 0.3, we need to prove that  $B_n$  belongs to  $\mathfrak{J}$ . We adapt the proof of Theorem 0.3 of [3], which relies on an argument of homotopy. We use the notations therein.

*Proof.* As a consequence of Stokes' formula, one can express

$$(u, v) \mapsto (u, v) \cdot \mathcal{F}(B_n) \tag{80}$$

as a weighted sum of bi-differential operators  $B_\Gamma$  for  $\Gamma \in G_{n,2}$  with weights

$$w'_\Gamma = \int_{Z_n} \omega'_\Gamma, \tag{81}$$

where  $Z_n$  is some subset of the boundary of the compactification  $\overline{C}_{n+2,0}^+$ , and  $\omega'_\Gamma$  is a differential form. The integral  $\int_{Z_n} \omega'_\Gamma$  is a sum of integrals over the top dimensional components of  $Z_n$ . Recall that  $\overline{C}_{n+2,0}^+$  corresponds to (possibly degenerate) configurations of  $n+2$  points in the Poincaré open half plane  $\mathfrak{H}$ . Top-dimensional components of the boundary correspond to configurations with exactly one cluster of points; writing  $\bar{1}, \bar{2}$  for the two distinguished points in the configuration, we observe that in  $Z_n$   $\bar{1}$  and  $\bar{2}$  remain in  $\mathfrak{H}$  and do not belong to the same cluster.

So, as in [3, Proof of Theorem 0.3], we have four types of components of  $Z_n$  to consider:

- one cluster at  $\bar{1}$ ;
- one cluster at  $\bar{2}$ ;
- one cluster in  $\mathfrak{H}$  not at  $\bar{1}$  or  $\bar{2}$ ;
- one cluster in  $\mathbb{R}$ .

The same argument as in [3], shows that type 3 and 4 components give 0 contribution, as well as any type 1 or 2 component with a cluster of more than two points at  $\bar{1}$  and  $\bar{2}$ .

We now show that the first type corresponds to a bi-differential operator of the form  $\mathcal{C}(\mathcal{I} \otimes \mathfrak{W}(\mathfrak{g}^*))$ , whereas the second corresponds to a bi-differential operator of the form  $\mathcal{C}(\mathfrak{W}(\mathfrak{g}^*) \otimes \mathcal{I})$ . Since here  $u, v$  are both invariant, case 2 is identical to case 1; so we shall only outline the argument for case 1. Let  $Z$  be a type 1 component of  $Z_n$ , with  $a$  the vertex infinitesimally close to  $\bar{1}$ . We know that  $Z$  is diffeomorphic to  $C_2 \times C_{n+1,0}$ . Let  $\Gamma$  be a graph in  $G_{n,2}$ . There has to be an edge from  $a$  to  $\bar{1}$ , otherwise the  $C_2$  contribution to  $\int_Z \omega'_\Gamma$  is 0. Similarly, there cannot be two edges with the same origin and ending at  $a$  and  $\bar{1}$ . Then  $\int_Z \omega'_\Gamma$  depends only on  $\bar{\Gamma}$  where  $\bar{\Gamma}$  is the graph obtained from  $\Gamma$  by collapsing the two vertices  $a, \bar{1}$ . Considering all graphs  $\Gamma'$  (with one edge from  $a$  to  $\bar{1}$ ), such that  $\bar{\Gamma}' = \bar{\Gamma}$  (there differ from  $\Gamma$  only by one edge ending at  $a$  or  $\bar{1}$ ), one sees that

$$(u, v) \cdot \mathcal{F} \left( \sum_{\Gamma', \bar{\Gamma}' = \bar{\Gamma}} B_{\Gamma'} \right)$$

is a sum of terms of the form

$$\sum_{j=1}^d ((u \cdot e_j^*) \partial([e_i, e_j]), v) \cdot \mathcal{F}(B_{\overline{\Gamma}}).$$

Since  $\sum_{j=1}^d (u \cdot e_j^*) \partial([e_i, e_j]) = u \cdot \text{adj}_{e_i}$ , by Lemma 4.7, the proof of Theorem 0.3 is complete.  $\square$

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