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Price-mediated trade with quantity signals: an axiomatic approach[☆]

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In Honour of Martin Shubik

Abstract

We consider a mechanism in which individuals send signals, indicating how much of each commodity they are willing to put up for trade. The mechanism produces prices and redistributes the commodities. We require that the map from signals to trades and prices, satisfy certain axioms and show that there are in essence only a finite number of mechanisms (i.e. maps) which satisfy these axioms. They include the Shapley mechanism and the Shapley–Shubik mechanism, and variants that lie “in between” the two. We also point out an open problem regarding a convexity property of these mechanisms, which is germane to the analysis of games based on them.

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1. Introduction

Suppose there are n individuals who wish to trade amongst themselves in m commodities. Two possible mechanisms¹ for accomplishing such trade have been suggested by [Shapley and Shubik \(1977\)](#), [Shapley \(1976\)](#), [Sahi and Yao \(1989\)](#). The purpose of this paper is to

[☆] A brief summary of the results in this paper was contained in a survey in [Dubey \(1994\)](#) but they are being presented here in full for the first time.

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¹ We should emphasize that by a mechanism we mean *only the rules of price formation and trade*. In this paper, we do not discuss the issues of what constrains or motivates the traders. Such issues can only be raised after the introduction of utilities, endowments, and solution concepts. Our concerns are much more primitive in that we seek only to address the possibility of axiomatizing mechanisms. Thus, the “commodities” in our model are not presumed to have intrinsic worth, and could correspond to fiat money, stock certificates, etc.

establish an axiomatic foundation for a class of mechanisms which includes both of these as special cases.

We start by describing the Shapley–Shubik and Shapley mechanisms.

In the Shapley–Shubik mechanism one of the commodities, say m , is called money, and plays a distinguished role. Each trader α sends a signal, which consists of a pair of numbers q_i^α, b_i^α for each commodity i , other than money. The first number q_i^α indicates how much of commodity i he is offering for sale, and b_i^α indicates how much money he is offering for the purchase of i . Having received this data, the mechanism does two things. First it computes a price for each commodity by the formula $p_i = (\sum_{\alpha=1}^n b_i^\alpha) / (\sum_{\alpha=1}^n q_i^\alpha)$; and then it redistributes the commodities so that α gets $r_m^\alpha = \sum_{i=1}^{m-1} p_i q_i^\alpha$ units of money and $r_i^\alpha = b_i^\alpha / p_i$ units of commodity i .

In the Shapley mechanism all commodities are treated symmetrically. Each trader α sends a signal which is an $m \times m$ matrix whose ij th entry a_{ij}^α indicates the amount of commodity i he is offering in exchange for commodity j . The mechanism calculates prices by solving the system of equations $\sum_{i=1}^m (\sum_{\alpha=1}^n a_{ij}^\alpha p_i) = p_j \sum_{i=1}^m (\sum_{\alpha=1}^n a_{ji}^\alpha)$; and then it redistributes the commodities, so that α gets $r_i^\alpha = p_i^{-1} \sum_{j=1}^m p_j a_{ji}^\alpha$ units of commodity i .

Both mechanisms are in the Cournot tradition, in that signals are denominated in quantities of commodities. In the Shapley–Shubik mechanism there are $m - 1$ *decentralized trading-posts* which are cleared independently of each other. On the other hand, in the Shapley mechanism the signals a_{ij} are addressed to a *window* for commodity j in a *central clearing-house*, which then determines prices and returns, based on all the signals.

The two mechanisms are special cases of what we would like to call a *G-mechanism*, based on a *completely reducible graph* G .

We say that a directed graph G is *irreducible*, if every node is connected to each of the other nodes by a *directed* path. We say that G is completely reducible if it is an (arc-)disjoint union of irreducible graphs.

Let G be a completely reducible, directed, graph with a node for each commodity $i = 1, \dots, m$. We define the *G-mechanism* as follows: each trader α sends a signal which consists of non-negative numbers a_{ij}^α for each arc (i, j) in G , where a_{ij}^α indicates the amount of commodity i that he is offering in exchange for commodity j . The prices and returns are then given by the formulas for the Shapley mechanism, where we understand a_{ij}^α to be 0 for non-existent arcs.

Clearly the Shapley mechanism corresponds to the complete graph, and it is easy to see that the Shapley–Shubik mechanism arises from the graph with arcs $(i, m), (m, i)$ for all $i < m$.

The price equations state that the total value of commodity i in the market equals the total value of all commodities that are “chasing” i . We now describe the conditions for the existence and uniqueness of a positive solution.

These conditions are discussed in Sahi and Yao (1989), and are somewhat subtle. Let us call an arc (i, j) *active* if $\sum_{\alpha} a_{ij}^\alpha$ is positive, and consider the subgraph of G spanned by the active arcs. Then the price equations have a positive solution if and only if this subgraph is completely reducible, and in this case the prices are uniquely determined up to independent, positive, scalar multiples on each irreducible component. This indeterminacy of prices has no effect on the returns.

On account of their explicit nature, the G -mechanisms may appear somewhat ad hoc. We will show in this paper that they are characterized by four axioms.

Consider an abstractly given mechanism which works as follows. Traders send out signals, which indicate how much of each commodity they are willing to put up for trade. The signal in any single commodity by a trader can, in general, be a vector.

After receiving all the signals, the mechanism determines prices and assigns “returns” to each trader. We require that the mechanism be efficient, i.e. that it redistribute everything it receives. We further require that for each trader, the values (under the prevailing prices) of his sales and purchases be equal.²

The first is an *aggregation* axiom. This says that if a trader pretends to be two persons, by splitting his signal, this has no effect on the prices or on the returns to the others.

The second axiom is an *invariance* requirement with respect to the units in which commodities are measured. Consider a change of units and re-denominate signals in them. The axiom says that the *physical* returns and prices remain unchanged, except that they are now quoted in the new units.

The third axiom is *price mediation*, and says that the returns that accrue to any trader depend, in an anonymous way, only upon his signal and the prevailing prices.

This is not to say that a trader does not influence prices by his signals; he invariably does, which is but to be expected in an oligopolistic set-up. The point is that any trader interacts with the others solely through the prices. Thus, prices mediate trade and summarize all the relevant information for any trader. In this sense they act as a “decoupling device”.

Finally, we have an *accessibility* axiom. Consider the “universal” set of return vectors that a trader can get as he varies his signals in a *single* commodity i , while others vary their signals arbitrarily. The axiom says that this set is closed. In other words, if the trader can get “arbitrarily close” to a particular return vector in this set, it is actually accessible!

Some alternatives to these axioms are discussed in [Section 5](#).

Our main theorem states that *any mechanism that satisfies the axioms is essentially a G -mechanism for some completely reducible graph G on the set of commodities*.

The reason for the qualification “essentially” is that the mechanism may have some “redundancies”. We describe here the prototypical example of a redundancy, deferring the precise definition to [Section 3](#).

Consider the Shapley mechanism with three commodities, and add an extra component “ a_{14} ” for the signals in commodity 1. Now define a new mechanism on the enlarged signal space as follows: first, shrink the new signals back to the “standard” form by the rule $(a_{11}, a_{12}, a_{13}, a_{14}) \mapsto (a_{11}, a_{12} + \lambda a_{14}, a_{13} + (1 - \lambda)a_{14})$, for some $0 \leq \lambda \leq 1$; and then apply the Shapley mechanism.

It is clear that the new mechanism satisfies all our axioms. However, it is equally clear that a trader who uses a new signal could achieve the same effect by sending the equivalent shrunk signal in the first three components and 0 in the fourth component for commodity 1. He uses the same amounts of each commodity in sending this signal, and leaves unchanged the prices and *everyone’s* returns! Thus, in this sense the additional component is redundant, and the new mechanism is essentially the same as the standard Shapley mechanism.

² Since the mechanism is efficient, the value equivalence is automatically true *on the aggregate*. Thus, the requirement is that the mechanism does not assign profitable trades to some traders at the expense of others.

In conclusion, let us point out that the real import of our paper is to show that the axioms *imply* the existence of *commodity markets*—the trading-posts of the Shapley–Shubik mechanism and the windows of the Shapley mechanism—and that the “abstract” signals are actually addressed to these specific markets.

2. The axioms

In this section we state our axioms more precisely after introducing the necessary notation.

A signal in commodity i is a non-negative vector with k_i components. Thus, the signal space for commodity i is $S[i] \equiv \mathbf{R}_+^{k_i}$; and the “full” signal space is the Cartesian product S of $S[1]$ through $S[m]$. S may be identified with the non-negative orthant \mathbf{R}_+^k where $k = k_1 + \dots + k_m$, and we will write S_+ for the positive orthant \mathbf{R}_{++}^k .

We will use i, j , for commodities, α, β , for traders and a, b for signals in S . Superscripts will refer to traders, and subscripts usually to commodities, except that the subscript l will be reserved for the components of a signal. We will write K_i for the set $\{k_1 + k_2 + \dots + k_{i-1} < l \leq k_1 + k_2 + \dots + k_i\}$. This is the set of components of a signal which are denominated in commodity i .

Let us write $C \equiv \mathbf{R}_+^m$ for the *commodity space*. There is a natural linear map $\chi : S \rightarrow C$, where the i th component of $\chi(a)$ is the sum of the K_i -components of a . Thus, $\chi(a)$ represents the commodity bundle required to send the signal a .

In matrix notation, we may write $\chi(a) = Aa$ where A is the $m \times k$ *auxiliary matrix* whose first k_1 columns are $(1, 0, \dots, 0)^t$, the next k_2 columns are $(0, 1, 0, \dots, 0)^t$, etc.

Prices p are to be thought of as (consistent) exchange-rates between commodities. Thus, they are naturally in the *price space* $P \equiv \mathbf{R}_{++}^m / \mathbf{R}_{++}$, the set of “rays” in \mathbf{R}_{++}^m . We will think of prices in P as vectors in \mathbf{R}_{++}^m , with the understanding that any notions involving them will be scale-invariant. Finally, *return vectors* are commodity bundles (in C) which are sent back to the traders by the mechanism, as a consequence of their collective choice of signals. We will use the letters p, q for prices and r, s for returns.

Let S^n be the n -fold Cartesian product of S with itself. An n -tuple of signals (a^1, \dots, a^n) in S^n represents a choice of signals by all agents, and will frequently be abbreviated as \mathbf{a} . It is inappropriate to assume that all such \mathbf{a} lead to price formation, but it *is* appropriate to require that this be true for signals which are “positive” on the aggregate. Thus, let

$$S(n) = \{\mathbf{a} \in S^n : a^1 + \dots + a^n \in S_+\}.$$

Definition. A market mechanism for m commodities is a collection of maps (one for each n) from $S(n)$ to $P \times C^n$ with the following two properties: suppose \mathbf{a} results in the price vector p and return vectors r^α , then

- (i) $\sum_{\alpha=1}^n \chi(a^\alpha) = \sum_{\alpha=1}^n r^\alpha$,
- (ii) $p \cdot \chi(a^\alpha) = p \cdot r^\alpha$ for $1 \leq \alpha \leq n$.

These are the efficiency and value-conservation requirements discussed previously. We are now ready for the precise statements of our axioms.

It is easier (and *sufficient!*) to state a weak form of the aggregation axiom, in which we only consider the effect of the “last” person splitting his signals.

Axiom 1 (Aggregation). Suppose $\mathbf{a} \in S(n)$ and $\mathbf{b} \in S(n + 1)$ are such that $a^\alpha = b^\alpha$ for $\alpha < n$ and $a^n = b^n + b^{n+1}$. Then \mathbf{a} and \mathbf{b} lead to the same prices; and if r and s are the return vectors then $r^\alpha = s^\alpha$ for $\alpha < n$.

The second axiom says that if we scale the units of commodity i by a positive scalar λ , the returns and prices remain the same except for their i th components, which are rescaled accordingly.

Axiom 2 (Invariance). Suppose $\mathbf{a}, \mathbf{b} \in S(n)$ are such that for all α , $b_\ell^\alpha = \lambda a_\ell^\alpha$ for $k_1 + \dots + k_{i-1} + 1 \leq \ell \leq k_1 + \dots + k_i$ and $b_i^\alpha = a_i^\alpha$ otherwise. Let p, q be the prices and let r^α, s^α be the returns resulting from \mathbf{a}, \mathbf{b} , respectively. Then $p_i = \lambda q_i$, and $s_i^\alpha = \lambda r_i^\alpha$ and the other components of the prices and return vectors are unchanged.

The next axiom captures the crucial, and anonymous, role played by prices in mediating trade.

Axiom 3 (Price mediation). Let r and s be the returns corresponding to $\mathbf{a}, \mathbf{b} \in S(n)$. Suppose that \mathbf{a} and \mathbf{b} lead to the same price vector, and that $a^\alpha = b^\beta$ for two traders α, β ; then r^α equals s^β .

It is easier (and sufficient) to state the last axiom for the special case of two traders. Let R_i be the set of return vectors for trader 1 as he varies his signals in $S[i]$, and trader 2 varies his signals in S_+ .

Axiom 4 (Accessibility). For each commodity i , the set R_i is closed.

3. The main results

To state our first result it is convenient to introduce the notion of a *product* of two mechanisms on *disjoint* commodity sets. This is the mechanism whose commodity set is the union; whose signal space is the Cartesian product; and whose price and return vectors are the *concatenations*, in the obvious manner.

Observe that we may modify the product mechanism by independently scaling the price vectors on the two commodity sets, and that this has no effect on the returns. We will *continue* to call such a modified mechanism a product of the two mechanisms.

An *irreducible* mechanism is one which is not a product of two smaller mechanisms.

If T is a non-negative $m \times m$ matrix, let $G(T)$ be the directed graph on m nodes which has an arc from i to j if the (i, j) th entry of T is positive. T will be called irreducible or completely reducible, if $G(T)$ has these properties.

If M is an $m \times k$ non-negative matrix, we will write $G(M)$ for $G(MA^t)$ where A is the $m \times k$ auxiliary matrix defined in Section 2, and t denotes transpose. The notions of

irreducibility and complete reducibility are defined in terms of $G(M)$ as above. We are now ready to state our first result

Theorem 1. *Any market mechanism satisfying the first three axioms is a product of irreducible mechanisms, each of which satisfies these axioms. Moreover, an irreducible mechanism satisfying these axioms determines, and is determined by, an irreducible, non-negative, column stochastic, $m \times k$ matrix M .*

For the explicit formulas in terms of M , we refer the reader forward to the proof of [Theorem 1](#).

To prepare for our second (and main) result, we now make precise the notion of a “redundant” component of a signal.

Consider a mechanism with signal space S , etc. as before. Fix a component l in K_i and let S'_+ be the set of signals in S whose l th component is 0, and all other components are strictly positive. Write $S'(n)$ for the set $\{a \in S^n : a^1 + \dots + a^n \in S'_+\}$, and suppose there is an extension of the mechanism from $S(n)$ to $S(n) \cup S'(n)$, which continues to satisfy the axioms.

In such a situation, we will say that l is *redundant* if, given any a in $S(n) \cup S'(n)$ and a trader α , there exists b in S'_+ such that: $\chi(b) = \chi(a^\alpha)$ and if α switches to the signal b , the prices and *everyone's returns* remain unchanged.

If a mechanism has a redundant component, then one gets a smaller mechanism by deleting l , i.e. by restricting the signal space to $S'(n)$. Iterating this procedure, one gets a mechanism with no redundant components. We will call this an *essential sub-mechanism*³ of the original mechanism.

Our main result is

Theorem 2. *Each mechanism satisfying the four axioms contains a unique essential sub-mechanism that is a G -mechanism, where $G = G(M)$ and M is as in [Theorem 1](#).*

4. Proofs

Fix a mechanism which satisfies the axioms of the previous section, and consider an n -tuple $a = (a^1, \dots, a^n)$ of signals in $S(n)$.

By repeated application of Axiom 1, we see that the price vector is the same as it would be if there were a *single* trader sending the *aggregate* signal $\bar{a} = a^1 + \dots + a^n$. Thus, the prices depend only on the aggregate signal. Combining this with [Axiom 3](#), we conclude that a trader's return vector is a function only of his signal and the aggregate signal!

Thus, the mechanism is specified by two functions, $\pi : S_+ \rightarrow P$, and $\rho : S \times S_+ \rightarrow C$, such that the signals a result in the prices $p = \pi(\bar{a})$ and yield the return vectors $r^\alpha = \rho(a^\alpha, \bar{a})$. Note that if a and b are signals in S and S_+ , respectively, then $\rho(a, b)$ is defined only if $b \geq a$.

³ It is clear that the “market game” based on an essential submechanism will have the same “Nash” allocations, and the same “ α -, β -cores”, etc. as that for the original mechanism.

Lemma 1. *The map ρ has a unique extension to $\mathbf{R}^k \times S_+$, that is linear in the first factor and homogeneous of degree zero in the second.*

Proof. By the efficiency of the mechanism, $\rho(a, b) \leq \chi(b)$, for all $a \leq b$; moreover if a and a' in S are such that $a + a'$ is less than b , then **Axiom 1** implies the functional (Cauchy) equation $\rho(a + a', b) = \rho(a, b) + \rho(a', b)$.

From Corollary 2 in **Aczel and Dhombres (1989, p. 35)**, we conclude that, for all non-negative λ and λ' such that $\lambda a + \lambda' a' \leq b$, we have

$$\rho(\lambda a + \lambda' a', b) = \lambda \rho(a, b) + \lambda' \rho(a', b). \tag{1}$$

Next let $a \leq b$ and choose $\lambda \geq 1$, then the argument just given shows that $\rho(\lambda a, \lambda b) = \lambda \rho(a, \lambda b)$. On the other hand, **Axiom 2** implies that the left side equals $\lambda \rho(a, b)$. Comparing these expressions we conclude that

$$\rho(a, b) = \rho(a, \lambda b). \tag{2}$$

Thus, even for a not less than b , we may define $\rho(a, b)$ via (2), by choosing λ sufficiently large. This extends ρ to all of $S \times S_+$; and the further extension to $\mathbf{R}^k \times S_+$ follows from (1). \square

Axiom 2 shows that the range of π is all of P , and so **Axiom 3** implies that there is a function $\tau : S \times P \rightarrow C$ such that $\rho(a, b) = \tau(a, \pi(b))$. It follows from **Lemma 1** that τ is linear in the first variable.

Let $\mathbf{1}$ be the m -dimensional vector of all 1's and let M be the $m \times k$ matrix such that $\tau(a, \mathbf{1}) = Ma$.

Axiom 2 implies that τ is “determined” by M in the following sense:

For p in \mathbf{R}^m , let D_p denote the $m \times m$ diagonal matrix $\text{diag}\{p_1, \dots, p_m\}$, and write E_p for the $k \times k$ “extended” diagonal matrix whose K_i -diagonal entries are all p_i ; then, as is easily checked,

$$\tau(a, p) = (D_p^{-1} M E_p) a. \tag{3}$$

Lemma 2. *The matrix M is non-negative and column stochastic.*

Proof. The non-negativity of M follows from that of τ .

Next, rewrite (3) as

$$D_p \tau(a^\alpha, p) = M E_p a^\alpha. \tag{4}$$

Then $\mathbf{1} \cdot (M E_p a^\alpha) = \mathbf{1} \cdot D_p \tau(a^\alpha, p) = p \cdot \tau(a^\alpha, p)$.

Recall the auxiliary matrix A defined in **Section 2**; then by the value conservation property, the last expression equals $p \cdot A a^\alpha$, which may be rewritten as $\mathbf{1} \cdot A E_p a^\alpha$. Since a^α is arbitrary, we get $\mathbf{1} M = \mathbf{1} A$, which implies the column stochasticity of M . \square

We show next that the price vector satisfies a system of linear equations involving M , A and the aggregate signal.

For b in S_+ , let D_b be the corresponding $k \times k$ diagonal matrix. Put $C_b = MD_bA^t$, $\Delta_b = AD_bA^t$, and $T_b = \Delta_b^{-1}C_b$. Thus, C_b is the $m \times m$ matrix whose i th column is a linear combination of the K_i -columns of M with coefficients given by the K_i -components of b . Since M is column stochastic, it follows that Δ_b is the diagonal matrix of the column sums of C_b , and thus we see that T_b is an $m \times m$, non-negative, column stochastic matrix.

Lemma 3. For a in $S(n)$, write $b = \bar{a}$, $p = \pi(b)$; then

$$T_b p = p. \tag{5}$$

Proof. Summing (4) over α we get $D_p \tau(b, p) = ME_p b$. By the efficiency of the mechanism we have $\tau(b, p) = Ab$, and combining this with the identity $AE_p = D_p A$ gives $(M - A)E_p b = 0$. Finally, using the identity $E_p b = D_b A^t p$, we get $(C_b - \Delta_b)p = 0$, and (5) follows. \square

Lemma 4. M is completely reducible.

Proof. Since T_b is column-stochastic it may be regarded as the transition matrix of a Markov process. Then (5) shows that p (after normalizing so that $\sum p_i = 1$) is a steady state probability distribution for T_b .

By assumption, for b in S_+ , (5), has a positive solution. Arguing as in Lemma 1 in Sahi and Yao (1989), we conclude that $G(T_b)$ must be completely reducible. (In Markovian terminology, this corresponds to the remark that such a process cannot contain any “transient” states.) Since $G(T_b) = G(M)$, the result follows. \square

Proof of Theorem 1. The complete reducibility of $G(M)$ implies that the set of nodes (commodities) can be partitioned in such a manner that $G(M)$ becomes an arc-disjoint union of irreducible subgraphs (on the subsets of the partition).

We will write I' and G' for a typical subset of I and its irreducible subgraph, and use “primes” to denote the restriction of various matrices and vectors to I' . Complete reducibility means that M is “block-diagonal” with blocks corresponding to the various M' . Thus, the equations (3) and (5) decompose over each I' to give

$$\tau(a', p') = (D_{p'}^{-1} M' E_{p'}) a', \tag{6}$$

$$T_{b'} p' = p'. \tag{7}$$

The irreducibility of G' implies that, up to a scalar multiple, p' is uniquely determined by (7), which we may rewrite as $C_{b'} p' = \Delta_{b'} p'$.

If G' has only 1 node, we set p' equal to 1, say; while if G' has more than 1 node, let p' be the vector whose components are the cofactors of the entries of the first row of $(\Delta_{b'} - C_{b'})$. It follows from Lemma 2 of Sahi and Yao (1989) that p' is a positive vector and that it satisfies (7).

Now using (6), we get a mechanism defined on each I' which is completely determined by the irreducible matrix M' . It is easy to check that each of these mechanisms satisfies the axioms, is irreducible, and that the original mechanism is their product. \square

Proof of Theorem 2. In view of [Theorem 1](#), we may restrict our attention to an irreducible mechanism with matrix M . Fix a commodity i , and let I be the set of commodities for which the corresponding entry is positive in at least *one* of the K_i -columns of M .

If R_i is as in [Axiom 4](#), it is easy to see from (3) that R_i is contained in the “face” $F_i = \{v \in C | v_j = 0 \text{ if } j \text{ is not in } I\}$. For each a in $S[i]$ and an arbitrary b in S_+ , we can choose λ so that $\lambda b - a$ is in S_+ . This shows that R_i contains $\{\rho(a, \lambda b - a) | a \in S[i], b \in S_+\}$.

Since $\pi(\lambda b) = \pi(b)$ and since the range of π is all of P , it follows that R_i contains the set $\{\tau(a, p) | a \in S[i], p \in P\}$. From (3) it follows that this set is the *interior* of F_i . [Axiom 4](#) now implies that R_i must be *equal* to F_i .

In particular, for each j in I it is possible to get a return that is positive *only* in its j th component. From (3) we see that this implies that for each such j , the j th *unit* vector must occur as one of the K_i -columns of M .

Given a signal b by trader 2, if a is any signal in $S[i]$ such that $a + b$ is in S_+ , it follows easily from (3) and (5) that the price and return vectors depend *only* on the linear combination of K_i weighted by a .

Since the K_i columns of M contain enough vectors to express each column as their convex combination, it follows that we may replace a by a signal a' which satisfies $\chi(a') = \chi(a)$, involves *only* these unit vectors and such that the prices and returns are unchanged.

This shows that if we restrict the mechanism to the unit vectors for each K_i we get an essential sub-mechanism. Moreover, this sub-mechanism is unique, except for the degenerate case in which the same unit vector occurs more than once among the K_i -columns of the original mechanism. □

5. Remarks

1. *Pre-price analysis:* Let us drop prices from the picture altogether, and think of the mechanism as only producing returns. Let [Axiom 2'](#) be the axiom obtained from [Axiom 2](#) after suppressing the price effects. Also instead of [Axiom 3](#), directly postulate the *anonymity* [Axiom 3'](#) that if any two signals are transposed, the output of the mechanism remains the same with the corresponding returns also transposed.

Arguing along the same lines as our proof of [Lemma 1](#), one obtains the following

Proposition. Any mechanism for which [Axioms 1, 2' and 3'](#) hold, determines, and is determined by, a continuous map $b \mapsto M_b$ from S_+ to the space of non-negative $m \times k$ matrices, satisfying (i) $M_b b = Ab$, and (ii) $M_{E_v b} = D_v M_b E_v^{-1}$, for all v in \mathbf{R}_{++}^m . Moreover, the correspondence between M_b and ρ is given by (iii) $\rho(a, b) = M_b a$.

(See [Section 2](#) and (3) for the definitions of A, D_v and E_v .)

2. *The value axiom (an alternative to invariance):* Let us weaken [Axiom 2](#) by restricting attention to the effect of *uniform* scaling in all commodities. In addition, assume directly that the range of π is all of P , and that the commodity-wise values of a trader's return vector depend only the component-wise values of his signal. In the notation of (3), this means that there is a function ϕ , say, such that $\phi(E_p a) = D_p \tau(a, p)$ for all a and p .

Then our results continue to hold. Indeed, replacing a by $E_p^{-1}a$, we get $\phi(a) = D_p\tau(E_p^{-1}a, p)$, from which it follows that ϕ is linear. Writing M for its matrix, we recover (3), and the rest of the arguments are unchanged.

3. *The maneuverability axiom* (an alternative to *accessibility*): Recall the map $\tau : S \times P \rightarrow C$, where $\tau(a, p)$ is the return to a trader if his signal is a , and the prices are p . (The existence of this map is equivalent to the price mediation axiom).

For any subset I of commodities, we define the corresponding *face* of $C = \mathbf{R}_+^m$ to be the set $\{v \in C : v_i > 0 \text{ for } i \in I, v_i = 0 \text{ for } i \notin I\}$. A subset of C will be called *quasi-open* if its intersection with each face is open in that face.

The maneuverability axiom is: for any $p \in P$ and each commodity i , the set $\{\tau(a, p) : a \in S[i]\}$ is quasi-open.

If we assume Axioms 1–3 and the maneuverability axiom, then [Theorem 2](#) continues to hold.

The map τ and the maneuverability axiom are best understood in the situation where we have a *continuum* of traders. Then prices remain unchanged as any single trader varies his signals. The basic idea behind this axiom is to ensure that the mechanism does not impose ad hoc limitations on trade. In other words, if an agent can obtain some positive returns by signals in a single commodity, then, by arbitrary variations of such signals, he should be able to maneuver those returns freely.

4. *Strong maneuverability and the Shapley mechanism*: Suppose we strengthen the maneuverability axiom by replacing the word “quasi-open” with “open in \mathbf{R}_+^m ”. It can easily be shown that this singles out the Shapley mechanism.
5. *Extensions of the domain*: Consider a mechanism (with its associated M), and an n -tuple of signals $\mathbf{a} = (a^1, \dots, a^n)$. Define the subgraph G of Digraph (M) by deleting arcs on which zero total weight is placed by \mathbf{a} . If G is completely reducible, then it is clear from our analysis that the mechanism can be extended continuously to \mathbf{a} . For other kinds of zeros in \mathbf{a} it is equally clear that it cannot be so extended, i.e. an “irremovable” discontinuity exists at \mathbf{a} . Various rules (like confiscation, or return, of commodities quoted by traders in \mathbf{a}) have been adopted to well-define the mechanism at such points ([Shapley and Shubik, 1977](#)).
6. *Restrictions on the mechanism*: First, suppose that commodity i is “exchangeable” for j in the mechanism, i.e. by signals denominated in i alone, a trader can obtain positive returns in j ; along with, perhaps, other commodities. (How much he gets of j will depend upon prices. Also notice, by the way, that through such signals, he affects not only the two prices p_i and p_j but, for many graphs, the *whole* vector p .) It is often natural to make a *symmetry* assumption on commodities: if i is exchangeable for j , then j is also exchangeable for i . With this, we can replace the directed graphs by undirected graphs.

Next one might suppose that, starting with only commodity i , if a trader engages in enough rounds of trade, then it should be possible to obtain positive amounts of j , for any $j \neq i$. In short, assume that the mechanism does not prohibit the *ultimate* conversion of i to j and, at most, repeated trading is needed to do so. If we postulate this *exchangeability* axiom, then we can restrict attention to irreducible graphs.

7. *Oligopolistic effect*: All of our mechanisms have the property that if the i -components of a signal are increased (keeping other components fixed), then this has the effect of

raising the prices of all the other commodities relative to i . This is a corollary of the following

Proposition. *Suppose b and b' in S_+ are such that each entry of column i of $C_{b'}$ is at least as large as the corresponding entry of C_b ; and all other entries in $C_{b'}$ and C_b are equal. Then, for all $j \neq i$*

$$\frac{\pi_j(b')}{\pi_i(b')} \geq \frac{\pi_j(b)}{\pi_i(b)}.$$

(See Section 4 for the definitions of π and C_b .)

Proof. Without loss of generality, let $i = 1$ and $j = 2$, and write B, B' for $\Delta_b - C_b, \Delta_{b'} - C_{b'}$. From Lemma 3 it follows that the cofactors of any row of B, B' give the prices $\pi(b), \pi(b')$. Let us choose row 1 and write p, p' for $\pi(b), \pi(b')$. Then

$$p_i = B_{1i}, p'_i = B'_{1i},$$

where B_{1i}, B'_{1i} are the cofactors of the 1 i th entry of B, B' .

Since B and B' differ only in the first column, we have

$$p_1 = B_{11} = B'_{11} = p'_1.$$

To prove the Proposition, we need to show that $B'_{12} \geq B_{12}$.

Let M and M' be the (1,2) minor matrices of B and B' . Then $B_{12} = -\det M, B'_{12} = -\det M'$. Notice that M and M' are identical, except for their first columns, where the entries of M' are at least as negative as the corresponding entries of M . Expanding $\det M$ and $\det M'$ along the first columns, it suffices to show that all the cofactors of the first column of M (which are the same as for M') are non-negative. This is a consequence of the following

Claim. *If E is a matrix with positive diagonal entries, non-positive off-diagonal entries, and non-negative column sums, then all cofactors of E are non-negative.*

Proof of Claim. First consider the case when the off-diagonal entries are negative and the column sums positive. By rescaling the columns if necessary, we may reduce to the case $E = I - F$, where F is a positive matrix, each whose column-sums is strictly less than one. Then the infinite series

$$I + F + F^2 + \dots,$$

converges to a positive matrix which is the inverse of E . Let E_{cof} denote the matrix whose ij th entry is the j th cofactor of E . Then

$$E^{-1} = (\det E)^{-1} E_{\text{cof}}.$$

It remains only to show that $\det E$ is positive. To see this, consider $f(\lambda) = \det(I - \lambda F)$. By the same argument as above $I - \lambda F$ is invertible, hence $f(\lambda) \neq 0$ for $0 \leq \lambda \leq 1$. Since $f(0) = 1 > 0$, $f(1)$ must be positive.

This establishes the claim for the special case. However, since the cofactors are continuous functions of the entries of the matrix, the general case follows by a simple limiting argument, verifying the claim. \square

The Proposition follows by observing that the (1,1) minor matrix of B satisfies the conditions of the claim, and the cofactors that we are interested in are those of this minor matrix.

8. *A question of convexity*: Finally, we describe an open problem for such mechanisms.

Take a mechanism that satisfies Axioms 1 to 4, and consider a trader with a fixed, finite endowment in the various commodities. For any *fixed* choice of signals by the others, his returns are a function of his signal alone. Let H be the set of final bundles which he can achieve by sending signals which *do not exceed* his endowment; and let \bar{H} be the comprehensive hull of H , i.e. $\bar{H} = \{x \in R_+^m : x \leq y \text{ for some } y \in H\}$.

Is \bar{H} always convex?

This is important for the Nash analysis of the strategic game based on the mechanism (see Dubey and Shubik, 1978; Amir et al., 1990; Sahi and Yao, 1989).

The Shapley–Shubik mechanism is discussed in Dubey and Shubik (1978); more generally, the case of a “tree” is analyzed in Amir et al. (1990); and, finally, the Shapley mechanism is treated in Sahi and Yao (1989). In each of these instances, the set \bar{H} is convex. The proof in Amir et al. (1990) and Sahi and Yao (1989) turns on the following simple fact: for fixed signals by the others, the final bundle of a trader can be computed from the prices alone, i.e. *his* signal affects his final bundle only via the prices!

Let \bar{P} be the set of prices that the trader can generate, given a fixed endowment and fixed signals by the others. The set \bar{P} is defined to be *geometrically convex* if, for any p, q in \bar{P} , the vector $(\sqrt{p_1q_1}, \dots, \sqrt{p_mq_m})$ is in \bar{P} . Using the fact above, it can be easily shown (as in Amir et al., 1990; Sahi and Yao, 1989) that

Geometric convexity of $\bar{P} \Rightarrow$ convexity of \bar{H} .

As shown in Amir et al. (1990) and Sahi and Yao (1989), \bar{P} is geometrically convex in the Shapley–Shubik and Shapley mechanisms. One might wonder whether this is so for all the mechanisms satisfying our axioms. Unfortunately this is false! The simplest instance when \bar{P} fails to be geometrically convex is for the mechanism with four commodities whose underlying (undirected graph) is “the square with one diagonal”.

However, numerical evidence from several computer trials carried out by G. Koren at SUNY Stony Brook, seems to indicate that \bar{H} is nevertheless convex. Thus, the question of the convexity of \bar{H} remains open.

9. *Intermediate G-mechanisms*: As was pointed out to us by Gael Giraud, the foreign exchange market may furnish a practical instance of a G -mechanism that lies in between the Shapley–Shubik and the Shapley mechanisms. No one currency can be singled out to play the role of money in that it, and it alone, is linked to other currencies. This rules out the Shapley–Shubik mechanism. On the other hand, not every pair of currencies can be directly exchanged for each other (e.g. Australian dollars do not trade against Hong Kong dollars on European markets). This rules out the Shapley mechanism also.
10. *The limit-price mechanism of Mertens*: Once again it was pointed out to us by Gael Giraud that a similar typology in terms of G -mechanisms can perhaps be embedded

in the framework of limit-price mechanisms *à la* Jean-François [Mertens \(this issue\)](#). It might “suffice” to let the quantity-signals depend upon limit-prices. Thus the axiomatization given here could provide a first step towards an axiomatization of limit-price mechanisms in terms of the graph of commodities for which limit-price orders can be sent to the market against each other.

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