A NEW FORMULA FOR WEIGHT MULTIPLICITIES AND CHARACTERS

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1. Introduction. The weight multiplicities of a representation of a simple Lie algebra $\mathfrak{g}$ are the dimensions of eigenspaces with respect to a Cartan subalgebra $\mathfrak{h}$. In this paper, we give a new formula for these multiplicities.

Our formula expresses the multiplicities as sums of positive rational numbers. Thus it is very different from the classical formulas of Freudenthal [F] and Kostant [Ks], which express them as sums of positive and negative integers. It is also quite different from recent formulas due to Lusztig [L1] and Littelmann [Li].

For example, for the multiplicity of the next-to-highest weight in the n-dimensional representation of $\mathfrak{sl}_2$, we get the following expression (which sums to 1):

$$\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(n-1)(n)}+\frac{1}{n}.$$

The key role in our formula is played by the *dual* affine Weyl group.

Let V_0, $(\ ,\ )$ be the real Euclidean space spanned by the root system R_0 of $\mathfrak{g}$, and let V be the space of affine linear functions on V_0. We identify V with $\mathbb{R}\delta\oplus V_0$ via the pairing $(r\delta+x,y)=r+(x,y)$ for $r\in\mathbb{R}$, $x, y\in V_0$.

The dual affine root system is $R=\{m\delta+\alpha^\vee \mid m\in\mathbb{Z},\ \alpha\in R_0\}\subseteq V$, where $\alpha^\vee$ means $2\alpha/(\alpha,\alpha)$ as usual. Fix a positive subsystem $R_0^+\subseteq R_0$ with base $\{\alpha_1,\dots,\alpha_n\}$, and let β be the highest *short* root. Then a base for R is given by $a_0=\delta-\beta^\vee$, $a_1=\alpha_1^\vee,\dots,a_n=\alpha_n^\vee$, and we write s_i for the (affine) reflection about the hyperplane $\{x \mid (a_i,x)=0\}\subseteq V_0$.

The dual affine Weyl group is the Coxeter group W generated by $s_0,\dots,s_n$, and the finite Weyl group is the subgroup W_0 generated by $s_1,\dots,s_n$. For $w\in W$, its *length* is the length of a reduced (i.e., shortest) expression of w in terms of the s_i. The group W acts on the weight lattice P of $\mathfrak{g}$, and each orbit contains a unique (minuscule) weight from the set

$$\mathbb{O}:=\left\{\lambda\in P \mid \left(\alpha^\vee,\lambda\right)=0 \text{ or } 1,\ \forall\alpha\in R_0^+\right\}.$$

Definition. For each λ in P, we define

(1) $\widetilde{\lambda}:=\lambda+(1/2)\sum_{\alpha\in R_0^+}\varepsilon_{(\alpha^\vee,\lambda)}\alpha$, where, for $t\in\mathbb{R}$, ε_t is 1 if $t>0$ and -1 if $t\le 0$;

Received 1 March 1999.
1991 *Mathematics Subject Classification*. Primary 17B10, 33C50; Secondary 20F55, 20E46.
Author's work supported by a National Science Foundation grant.

(2) $w_\lambda :=$ unique shortest element in W such that $\overline{\lambda} := w_\lambda \cdot \lambda \in \mathbb{C}$.

We fix a reduced expression $s_{i_1} \cdots s_{i_m}$ for w_λ, and, for each $J \subseteq \{1, \ldots, m\}$, we define

(3) $w_J :=$ the element of W obtained by *deleting* s_{i_j}, $j \in J$, from the product $s_{i_1} \cdots s_{i_m}$;

(4) $c_J := \prod_{j \in J} c_j$, where $c_j := (a_{i_j}, \widetilde{\lambda_{(j)}})^{-1}$ and $\lambda_{(j)} := s_{i_{j-1}} \cdots s_{i_1} \cdot \overline{\lambda}$.

Let $P^+ \subset P$ be the cone of dominant weights; and, for $\lambda \in P^+$, let V_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ.

THEOREM 1.1. *For λ in P^+ and μ in P, the multiplicity $m_\lambda(\mu)$ of μ in V_λ is given by $m_\lambda(\mu) := (|W_0 \cdot \lambda| / |W_0 \cdot \mu|) \sum_J c_J$, where the summation is over all J such that $w_J^{-1} \cdot \overline{\lambda}$ is in $W_0 \cdot \mu$.*

(We prove in Corollary 6.2 that the c_J's are positive.)

For μ in P, let e^μ denote the function $x \mapsto e^{(\mu, x)}$ on V_0. Then W acts on the e^μ's by virtue of its action on P, that is, $s_i e^\mu = e^{s_i \cdot \mu}$, and Theorem 1.1 is equivalent to the following formula for the character $\chi_\lambda := \sum_\mu m_\lambda(\mu) e^\mu$ of V_λ.

THEOREM 1.2. *We have $\chi_\lambda = (|W_0 \cdot \lambda| / |W_0|) \sum_{w \in W_0} w (s_{i_m} + c_m) \cdots (s_{i_1} + c_1) e^{\overline{\lambda}}$.*

We obtain Theorem 1.2 as a consequence of a more general result, namely, an analogous formula for the generalized Jacobi polynomial P_λ of Heckman and Opdam. For the definition and properties of P_λ, we refer the reader to [HSc] and [O]. We recall here that P_λ depends on certain parameters k_α, $\alpha \in R_0$, such that $k_{w \cdot \alpha} = k_\alpha$ for all $w \in W_0$. For special values of k_α, P_λ can be interpreted as a spherical function on a compact symmetric space. In particular, in the limit as all $k_\alpha \to 1$, we have $P_\lambda \to \chi_\lambda$.

Definition. In the context of the previous definition, for λ in P, we *redefine*

(1′) $\widetilde{\lambda} := \lambda + (1/2) \sum_{\alpha \in R_0^+} k_\alpha \varepsilon_{(\alpha^\vee, \lambda)} \alpha$;

(4′) $c_j = k_{i_j} (a_{i_j}, \widetilde{\lambda_{(j)}})^{-1}$, where $k_0 = k_\beta$ and $k_i = k_{\alpha_i}$ for $i \geq 1$.

THEOREM 1.3. *For λ in P^+ and for c_j as above, the Heckman-Opdam polynomial P_λ is given by the same formula as in Theorem 1.2.*

For λ in P^+, define $c_\lambda := (|W_0| / |W_0 \cdot \lambda|) \prod_j (a_{i_j}, \widetilde{\lambda_{(j)}})$, and let $\mathcal{P} := \mathbb{Z}_+[k_\alpha]$ be the set of polynomials in the parameters k_α with nonnegative integral coefficients. Then we prove the following theorem.

THEOREM 1.4. *We have that c_λ is in $\mathcal{P}$, as are all coefficients of $c_\lambda P_\lambda$.*

Theorem 1.4 is a generalization of the main result of [KS] to arbitrary root systems. Our proof depends on three fundamental ideas in the "new" theory of special functions.

The first idea, due to Macdonald, Heckman, Opdam, and others, is that one can treat root multiplicities on a symmetric space as parameters.

The second idea, due to Dunkl and Cherednik, is that radial parts of invariant differential operators on symmetric spaces can be written as polynomials in certain commuting first-order differential-reflection operators, namely, the Cherednik operators.

The third idea is the method of intertwiners for Cherednik operators. This was developed in [KS], [K], [S1], and [C2], and it can be regarded as the double affine analog of Lusztig's fundamental relation [L2] in the affine Hecke algebra.

Using the intertwiners of [C2] and [S2], our results can be extended to the context of Macdonald polynomials and to the 6-parameter Koornwinder polynomials. These intertwiners correspond to the affine Weyl group (rather than the dual affine Weyl group) and hence are *not* appropriate for the present context. We shall discuss them elsewhere in [S3].

2. Preliminaries. The results of this section are due to Cherednik [C1], Heckman, and Opdam [O].

Let $\mathbb{F} = \mathbb{R}(k_\alpha)$ be the field of rational functions in the parameters k_α, and let $\mathcal{R}$ be the $\mathbb{F}$-span of $\{e^\lambda \mid \lambda \in P\}$ regarded as a W-module.

Definition. For $y \in V_0$, the Cherednik operator D_y is defined by

$$D_y = \partial_y + \sum_{\alpha \in R_0^+} (y,\alpha) k_\alpha \frac{1}{1-e^{-\alpha}}(1-s_\alpha) - (y,\rho), \quad \text{where } \rho := \frac{1}{2}\sum_{\alpha \in R_0^+} k_\alpha \alpha.$$

Here are some basic facts about Cherednik operators from [O, Section 2].

PROPOSITION 2.1. *We have the following.*
(1) *The operators D_y act on $\mathcal{R}$ and commute pairwise.*
(2) *For $i = 1, \ldots, n$, we have $s_i D_y - D_{s_i y} s_i = -k_i(y, \alpha_i)$.*
(3) *There is a basis $\{E_\lambda \mid \lambda \in P\}$ of $\mathcal{R}$, characterized uniquely as follows:*
  (a) *the coefficient of e^λ in E_λ is 1;*
  (b) *$D_y E_\lambda = (y, \widetilde{\lambda}) E_\lambda$, where $\widetilde{\lambda}$ is as in Definition (1') of the introduction.*
(4) *For λ in P^+, the Heckman-Opdam polynomial P_λ equals $(|W_0 \cdot \lambda| / |W_0|) \sum_{w \in W_0} w E_\lambda$.*
(5) *For $i = 1, \ldots, n$, if $s_i \cdot \lambda \neq \lambda$, then $\widetilde{s_i \cdot \lambda} = s_i \cdot \widetilde{\lambda}$.*

3. The affine reflection. In this section, we prove some basic properties of the affine reflection s_0.

LEMMA 3.1. *If α is a positive root different from β, then $(\alpha^\vee, \beta)$ equals zero or* 1.

Proof. Since β is in P^+, $(\alpha^\vee, \beta)$ is a nonnegative integer. Also, since β is a short root, we have $(\alpha, \alpha) \geq (\beta, \beta)$. So, by the Cauchy-Schwartz inequality, we get

$$\big(\alpha^\vee, \beta\big) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \leq 2\frac{(\alpha,\beta)}{(\alpha,\alpha)^{1/2}(\beta,\beta)^{1/2}} \leq 2.$$

If $\alpha \neq \beta$, then α is not proportional to β and the last inequality is strict. □

For $i = 0, 1, 2$, define $R_0^i = \{\alpha \in R_0^+ \mid (\alpha^\vee, \beta) = i\}$, and, for α in R_0^+, put

$$\alpha' = \begin{cases} s_\beta \cdot \alpha & \text{if } \alpha \in R_0^0, \\ -s_\beta \cdot \alpha & \text{if } \alpha \in R_0^1 \cup R_0^2. \end{cases}$$

LEMMA 3.2. *The involution $\alpha \mapsto \alpha'$ acts trivially on R_0^0 and R_0^2, and permutes R_0^1.*

Proof. For α in R_0^1, we have $(\alpha'^\vee, \beta) = (\alpha^\vee, -s_\beta \cdot \beta) = (\alpha^\vee, \beta) = 1$, which implies that α' is a (positive) root in R_0^1. The assertions about R_0^0 and $R_0^2 = \{\beta\}$ are obvious. □

LEMMA 3.3. *For λ in P, if $s_0 \cdot \lambda \neq \lambda$, then $\widetilde{s_0 \cdot \lambda} = s_0 \cdot \widetilde{\lambda}$.*

Proof. We compute $s_0 \cdot \widetilde{\lambda} = \beta + s_\beta \widetilde{\lambda}$ using Lemma 3.2 and $k_\alpha = k_{\alpha'}$. This gives

$$s_0 \cdot \widetilde{\lambda} = \beta + s_\beta \cdot \lambda + \frac{1}{2} \sum_{\alpha \in R_0^0} k_\alpha \varepsilon_{(\alpha^\vee, \lambda)} \alpha - \frac{1}{2} \sum_{\alpha \in R_0^1 \cup R_0^2} k_\alpha \varepsilon_{(\alpha'^\vee, \lambda)} \alpha.$$

Comparing this to the expression for $\widetilde{\mu}$ with $\mu = s_0 \cdot \lambda$, it suffices to show that

$$\varepsilon_{(\alpha^\vee, \mu)} = \begin{cases} \varepsilon_{(\alpha^\vee, \lambda)} & \text{if } \alpha \in R_0^0, \\ -\varepsilon_{(\alpha'^\vee, \lambda)} & \text{if } \alpha \in R_0^1 \cup R_0^2. \end{cases}$$

For α in R_0^0, we easily compute that $(\alpha^\vee, \mu) = (\alpha^\vee, \lambda)$.

For α in R_0^1, we get $(\alpha^\vee, \mu) = (\alpha^\vee, \beta + s_\beta \cdot \lambda) = 1 - (\alpha'^\vee, \lambda)$. Being an integer, $(\alpha'^\vee, \lambda)$ is either less than or equal to zero or greater than or equal to 1. In either case, we get $\varepsilon_{(\alpha^\vee, \mu)} = -\varepsilon_{(\alpha'^\vee, \lambda)}$.

Finally, for α in R_0^2, we have $\alpha = \alpha' = \beta$ and $(\beta^\vee, \mu) = 2 - (\beta^\vee, \lambda)$. Now $s_0 \lambda \neq \lambda$ implies that $(\beta^\vee, \lambda) \neq 1$; thus we have either $(\beta^\vee, \lambda) \geq 2$ or $(\beta^\vee, \lambda) \leq 0$. In either case, we get $\varepsilon_{(\beta^\vee, \lambda)} = \varepsilon_{(\beta^\vee, \lambda)} = -\varepsilon_{(\beta^\vee, \mu)}$. □

4. The intertwining relation. Dualizing the action $y \mapsto w \cdot y$ of W on V_0, we get a representation $v \mapsto wv$ of W on V satisfying $(wv, y) = (v, w^{-1} \cdot y)$. For y in V_0 and w in W_0, we have $wy = w \cdot y$. The affine reflection s_0 acts on V by

$$s_0(r\delta + y) = (y, \beta)\delta + r\delta + s_\beta y.$$

For $v = r\delta + y$ in V, we define the *affine* Cherednik operator simply by putting $D_v = D_y + rI$, where I is the identity operator. From Proposition 2.1(2), we know the intertwining relations between the (affine) Cherednik operators and $s_1, \dots, s_n$. In this section, we prove the following intertwining relation between these operators and s_0.

PROPOSITION 4.1. *For $v = r\delta + y$ in V, we have $D_v s_0 - s_0 D_{s_0 v} = k_\beta (y, \beta)$.*

Proof. Let us write N_α for $1/(1-e^{-\alpha})(1-s_\alpha)$, so that

$$D_v = \partial_y + \sum k_\alpha(y,\alpha)N_\alpha - (y,\rho) + r.$$

Since $s_\beta N_\alpha = N_{s_\beta\cdot\alpha} s_\beta$ and $s_\beta \partial_y = \partial_{s_\beta y} s_\beta$, we get

$$s_\beta D_v s_\beta = \partial_{s_\beta y} + \sum_{\alpha\in R_0^+} k_\alpha(y,\alpha)N_{s_\beta\cdot\alpha} - (y,\rho) + r.$$

Now, partitioning $R_0^+ = R_0^0 \cup R_0^1 \cup R_0^2$ and using Lemma 3.2, we get

$$s_\beta D_v s_\beta = \partial_{s_\beta y} + \sum_{\alpha\in R_0^0} k_\alpha\big(s_\beta y,\alpha\big)N_\alpha - \sum_{\alpha\in R_0^1\cup R_0^2} k_\alpha\big(s_\beta y,\alpha\big)N_{-\alpha} - (y,\rho) + r.$$

The following identities are easy to check:

(1) $e^\beta \partial_{s_\beta y} e^{-\beta} = \partial_{s_\beta y} + (y,\beta)$;

(2) $e^\beta N_\alpha e^{-\beta} = N_\alpha$ for $\alpha \in R_0^0$;

(3) $e^\beta N_{-\alpha} e^{-\beta} = 1 - N_\alpha$ for $\alpha \in R_0^1$;

(4) $e^\beta N_{-\beta} e^{-\beta} = 1 - N_\beta + s_0$.

Using these, we get the following formula for $s_0 D_v s_0 = e^\beta (s_\beta D_v s_\beta) e^{-\beta}$:

$$\partial_{s_\beta y} + (y,\beta) + \sum_{\alpha\in R_0^+} k_\alpha\big(s_\beta y,\alpha\big)N_\alpha - \sum_{\alpha\in R_0^1\cup R_0^2} k_\alpha\big(s_\beta y,\alpha\big) - k_\beta\big(s_\beta y,\beta\big)s_0 - (y,\rho) + r.$$

Since $\sum_{\alpha\in R_0^1\cup R_0^2} k_\alpha(s_\beta y,\alpha) = (s_\beta y, \rho - s_\beta\cdot\rho) = (s_\beta y,\rho) - (y,\rho)$, we get

$$s_0 D_v s_0 = D_{s_\beta y} + (y,\beta) - k_\beta(s_\beta y,\beta)s_0 + r = D_{s_0 v} + k_\beta(y,\beta)s_0.$$

The result follows. □

5. The Heckman-Opdam polynomials. Let E_λ be as in Proposition 2.1.

PROPOSITION 5.1. *The polynomials E_λ satisfy the following recursions:*

(1) *$E_\lambda = e^\lambda$ for $\lambda \in \mathbb{O}$;*

(2) *if $s_i\cdot\lambda \neq \lambda$, then $(s_i + (k_i/(a_i,\widetilde{\lambda})))E_\lambda$ is a multiple of $E_{s_i\cdot\lambda}$.*

Proof. For (1), we check simply that $D_y e^\lambda = (y,\widetilde{\lambda})e^\lambda$, using the identity

$$N_\alpha e^\lambda = \begin{cases} e^\lambda & \text{if } \big(\alpha^\vee,\lambda\big) = 1, \\ 0 & \text{if } \big(\alpha^\vee,\lambda\big) = 0. \end{cases}$$

For (2), we write F for $(s_i + (k_i/(a_i,\widetilde{\lambda})))E_\lambda$ and first consider $i \neq 0$. Then, for y in V_0, using Proposition 2.1(2), we get

$$D_y F = \left(s_i D_{s_i y} - k_i(y,\alpha_i) + \frac{k_i}{\big(a_i,\widetilde{\lambda}\big)} D_y\right)E_\lambda = \left(\big(s_i y,\widetilde{\lambda}\big)s_i + k_i\frac{\big(y,\widetilde{\lambda}\big)}{\big(a_i,\widetilde{\lambda}\big)} - k_i(y,\alpha_i)\right)E_\lambda.$$

Since $(y,\widetilde{\lambda})-(y,\alpha_i)(a_i,\widetilde{\lambda})=(s_i y,\widetilde{\lambda})$, using Proposition 2.1(5), we get

$$D_y F=\big(s_i y,\widetilde{\lambda}\big)F=\big(y,s_i\cdot\widetilde{\lambda}\big)F=\big(y,\widetilde{s_i\cdot\lambda}\big)F.$$

This proves (2) for $i\neq 0$. For $i=0$, we use Proposition 4.1 to get

$$D_y F=\left(s_0 D_{s_0 y}+k_\beta(y,\beta)+\frac{k_\beta}{(a_0,\widetilde{\lambda})}D_y\right)E_\lambda=\left(\big(s_0 y,\widetilde{\lambda}\big)s_0+k_\beta\frac{(y,\widetilde{\lambda})}{(a_0,\widetilde{\lambda})}+k_\beta(y,\beta)\right)E_\lambda.$$

This time, using $(y,\widetilde{\lambda})+(a_0,\widetilde{\lambda})(y,\beta)=(s_0 y,\widetilde{\lambda})$ and Lemma 3.3, we get

$$D_y F=\big(s_0 y,\widetilde{\lambda}\big)F=\big(y,s_0\cdot\widetilde{\lambda}\big)F=\big(y,\widetilde{s_0\cdot\lambda}\big)F.$$

This completes the proof of (2) for $i=0$. □

COROLLARY 5.2. *For λ in P, and c_i as in Definition (4′) of the introduction, we have*

$$E_\lambda=\big(s_{i_m}+c_m\big)\cdots\big(s_{i_1}+c_1\big)e^{\overline{\lambda}}.$$

Proof. By the minimality of w_λ, if w is a *proper* subexpression of $w_\lambda^{-1}=s_{i_m}\cdots s_{i_1}$, then $w\cdot\overline{\lambda}\neq\lambda$. This means that the coefficient of e^λ in $(s_{i_m}+c_m)\cdots(s_{i_1}+c_1)e^{\overline{\lambda}}$ is 1. The result now follows from Proposition 5.1. □

Proof of Theorem 1.3. This follows from Corollary 5.2 and Proposition 2.1(4). □

6. Positivity. Let $\mathcal{P}_1\subset\mathcal{P}$ be the set of polynomials of degree less than or equal to 1, with nonnegative integral coefficients and a *positive* constant term. For λ in P, let a_{i_j} and $\widetilde{\lambda_{(j)}}$ be as in Definition (4′) of the introduction.

PROPOSITION 6.1. *For each $j=1,\dots,m$, $(a_{i_j},\widetilde{\lambda_{(j)}})$ belongs to $\mathcal{P}_1$.*

Proof. Fix j and write $\mu=\lambda_{(j)}$, $i=i_j$, and $w=s_{i_1}\cdots s_{i_{j-1}}$. We need to show that $(a_i,\widetilde{\mu})$ has a positive constant term and nonnegative integral coefficients.

The lengths of w and ws_i must be $j-1$ and j, respectively, since otherwise we could shorten the expression $s_{i_1}\cdots s_{i_m}$ for w_λ. By a standard argument (see [Hu, Chapter 5]), this implies that $w(a_i)$ is a positive (affine) coroot in R^+. Since $\overline{\lambda}=\overline{\mu}$ is minuscule, we conclude that

$$0\leq\big(w(a_i),\overline{\mu}\big)=\big(a_i,w^{-1}\cdot\overline{\mu}\big)=(a_i,\mu).$$

If (a_i,μ) were zero, then $\lambda_{(j+1)}=s_i\cdot\mu=\mu=\lambda_{(j)}$ and we could shorten the expression for w_λ by dropping s_{i_j}. This shows that (a_i,μ), which is the constant term of $(a_i,\widetilde{\mu})$, is positive.

If $i=0$, the nonconstant part of $(a_0,\widetilde{\mu})$ is

$$-\frac{1}{2}\sum_{\alpha\in R_0^+}k_\alpha\varepsilon_{(\alpha^\vee,\mu)}\big(\beta^\vee,\alpha\big),$$

and we consider separately the contributions of R_0^0, R_0^1, and R_0^2.

For α in R_0^0, the contribution is zero.

For $\alpha = \beta$ in R_0^2, we get the term $-\varepsilon_{(\beta^\vee,\mu)} k_\beta$. By the first part, (a_0, μ) is a positive integer. Hence $(\beta^\vee, \mu) = 1 - (a_0, \mu) \leq 0$, which implies that $-\varepsilon_{(\beta^\vee,\mu)} = 1$.

The roots in R_0^1 can be grouped in pairs $\{\alpha, -s_\beta \cdot \alpha\}$, and the contribution of such a pair is

$$-k_\alpha \frac{\varepsilon_{(\alpha^\vee,\mu)} + \varepsilon_{(-s_\beta\alpha^\vee,\mu)}}{2} \big(\beta^\vee, \alpha\big).$$

Now $(\beta^\vee, \alpha)$ is positive, so the coefficient above is a nonnegative integer, unless $(\alpha^\vee, \mu)$ and $(-s_\beta\alpha^\vee, \mu)$ are both greater than zero. But in this case, we would get

$$0 < \big(\alpha^\vee, \mu\big) - \big(s_\beta\alpha^\vee, \mu\big) = \big(\alpha^\vee, \mu - s_\beta \cdot \mu\big) = \big(\beta^\vee, \mu\big)\big(\alpha^\vee, \beta\big) \leq 0,$$

which is a contradiction.

The argument is similar if $i > 0$. The nonconstant part of $(a_i, \widetilde{\mu})$ is

$$\frac{1}{2} \sum_{\alpha \in R_0^+} k_\alpha \varepsilon_{(\alpha^\vee,\mu)} (a_i, \alpha).$$

To compute this, we divide R_0^+ into three disjoint sets consisting of $\{\alpha_i\}$, {the roots orthogonal to α_i}, and {the remaining positive roots}. For $\alpha = \alpha_i$, we get the coefficient $\varepsilon_{(a_i,\mu)}$, which is 1 since $(a_i, \mu) > 0$ by the first part. If α is orthogonal to α_i, then the coefficient is zero. Finally, the remaining positive roots can be grouped into pairs $\{\alpha, s_i \cdot \alpha\}$, where we may assume that $(\alpha^\vee, \alpha_i) > 0$. The contribution of each such pair is

$$k_\alpha \frac{\varepsilon_{(\alpha^\vee,\mu)} - \varepsilon_{(s_i\alpha^\vee,\mu)}}{2} (a_i, \alpha).$$

Now $(\alpha^\vee, \alpha_i) > 0$ implies $(a_i, \alpha) > 0$. Therefore, this coefficient is a nonnegative integer, unless $(\alpha^\vee, \mu) \leq 0$ and $(s_i\alpha^\vee, \mu) > 0$. But if this were the case, then we would have

$$0 > \big(\alpha^\vee, \mu\big) - \big(s_i\alpha^\vee, \mu\big) = \big(\alpha^\vee, \mu - s_i \cdot \mu\big) = (a_i, \mu)\big(\alpha^\vee, \alpha_i\big) > 0,$$

which is a contradiction. □

Proof of Theorem 1.4. This follows from Theorem 1.3 and Proposition 6.1. □

Setting all the k_α's equal to 1 in Proposition 6.1, we deduce the following corollary.

COROLLARY 6.2. *The constants c_j and c_J in Theorems 1.1 and 1.2 are positive.*

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