

On an analytic description of the α -cosine transform on real Grassmannians.

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Abstract

The goal of this paper is to describe the α -cosine transform on functions on real Grassmannian $Gr_i(\mathbb{R}^n)$ in analytic terms as explicitly as possible. We show that for all but finitely many complex α

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the α -cosine transform is a composition of the $(\alpha + 2)$ -cosine transform with an explicitly written (though complicated) $O(n)$ -invariant differential operator. For all exceptional values of α except one we interpret the α -cosine transform explicitly as either the Radon transform or composition of two Radon transforms. Explicit interpretation of the transform corresponding to the last remaining value α , which is $-(\min\{i, n - i\} + 1)$, is still an open problem.

Contents

1	Introduction	2
2	Some linear algebraic isomorphisms.	7
3	Invariant form of the α-cosine transform.	10
4	Invariant form of the Radon transform.	11
5	Composition of Radon transforms	13
6	The precise statement and proof of Theorem 1.3.	14
7	Support of the distributional kernel of the α-cosine transform.	27
A	Appendix. Proof of Theorem 6.12.	34

1 Introduction

The goal of this paper is to describe the α -cosine transform on functions on real Grassmannians in analytic terms as explicitly as possible. This goal has been achieved in this paper to a certain extent, though still there are questions requiring further clarification.

To formulate the problem more precisely, we have to remind the definition of the α -cosine transform. Fix an n -dimensional Euclidean space V . Let $1 \leq i \leq n - 1$ be an integer. Let $Gr_i(V)$ denote the Grassmannian of i -dimensional linear subspaces. For a pair of subspaces $E, F \in Gr_i(V)$ one defines (the absolute value of) the cosine of the angle $|\cos(E, F)|$ between

E and F as the coefficient of the distortion of measure under the orthogonal projection from E to F . More precisely, let $q: E \rightarrow F$ denote the restriction to E of the orthogonal projection $V \rightarrow F$. Let $A \subset E$ be an arbitrary subset of finite positive Lebesgue measure. Then define

$$|\cos(E, F)| := \frac{\text{vol}_F(q(A))}{\text{vol}_E(A)},$$

where $\text{vol}_F, \text{vol}_E$ are Lebesgue measures on F, E respectively induced by the Euclidean metric on V normalized so that the Lebesgue measures of unit cubes are equal to 1. It is easy to see that $|\cos(E, F)|$ is independent of the set A and is symmetric with respect to E and F .

1.1 Example. If $i = 1$ then $|\cos(E, F)|$ is the usual cosine of the angle between two lines.

For $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) \geq 0$ let us define the α -cosine transform

$$T_\alpha: C^\infty(\text{Gr}_i(V)) \rightarrow C^\infty(\text{Gr}_i(V))$$

by

$$(T_\alpha f)(E) = \int_{F \in \text{Gr}_i(V)} |\cos(E, F)|^\alpha f(F) dF, \quad (1.1)$$

where dF is the $O(n)$ -invariant Haar probability measure on $\text{Gr}_i(V)$.

1.2 Remark. One can show that the integral (1.1) absolutely converges for $\text{Re}(\alpha) > -1$ (see [6, Lemma 2.1]).

It is well known (see below) that T_α has a meromorphic continuation in $\alpha \in \mathbb{C}$. For $\alpha_0 \in \mathbb{C}$ we will denote by S_{α_0} the first non-zero coefficient in the decomposition of the meromorphic function T_α near α_0 , namely

$$T_\alpha = (\alpha - \alpha_0)^k \cdot (S_{\alpha_0} + O(\alpha - \alpha_0)) \text{ as } \alpha \rightarrow \alpha_0, S_{\alpha_0} \neq 0.$$

Thus

$$S_{\alpha_0}: C^\infty(\text{Gr}_i(V)) \rightarrow C^\infty(\text{Gr}_i(V))$$

is defined and non-zero for any $\alpha_0 \in \mathbb{C}$, and coincides with T_{α_0} for all but countably many α_0 (necessarily $\text{Re}(\alpha_0) \leq -1$ for exceptional values of α_0). S_α will also be called the α -cosine transform.

The α -cosine transform on Grassmannians, including the case of even functions on the sphere (e.g. $i = 1$), was studied by both analysts and geometers for a long period of time: [21, 24, 25, 26, 27, 28, 29]. The case $\alpha = 1$ plays a special role in convex and stochastic geometry, see [11, 12, 13], and in particular in valuation theory [2, 3, 4, 5].

Obviously the α -cosine transform S_α (and also T_α) commutes with the natural action of the orthogonal group $O(n)$ on functions on the Grassmannian. In representation theory there is a detailed information available about the action of $O(n)$ and $SO(n)$ on functions on Grassmannians (see Section 6 below). This allows one to apply tools from the representation theory of $O(n)$ and $SO(n)$ to the problem. This method was used in [11, 12, 13] in the case $\alpha = 1$.

It was observed in [2] for $\alpha = 1$ and in [6] for general α that the α -cosine transform can be rewritten in such a way to commute with an action of the much larger full linear group $GL(V)$. In that language it is a map

$$S_\alpha: C^\infty(Gr_i(V), L_\alpha) \rightarrow C^\infty(Gr_{n-i}(V), M_\alpha),$$

where L_α, M_α are $GL(V)$ -equivariant line bundles over Grassmannians, and a choice of a Euclidean metric on V induces identification $Gr_{n-i}(V) \simeq Gr_i(V)$ by taking the orthogonal complement, and $O(n)$ -equivariant identifications of L_α and M_α with trivial line bundles. For the explicit description of L_α, M_α see (3.3)-(3.4) in Section 3 below. Moreover it was observed in [6] that the α -cosine transform is essentially a special case of the well known representation theoretical construction of intertwining integrals (see e.g. [31]). These observations opened a way to use the infinite dimensional representation theory of $GL(V)$ in the study of the α -cosine transform. Indeed in some more recent studies of the α -cosine transform [32, 23, 16] the representation theory of $GL(V)$ played the key role.

Very recently the second and the third authors [15] have proven the following new representation theoretical characterization of the α -cosine and Radon transforms. Let L and M be $GL(V)$ -equivariant complex line bundles over real Grassmannians $Gr_i(V)$ and $Gr_j(V)$ respectively. Then the

space of $GL(V)$ -equivariant linear continuous maps

$$C^\infty(Gr_i(V), L) \rightarrow C^\infty(Gr_j(V), M)$$

is at most 1-dimensional. Moreover if L and M are $O(n)$ -equivariantly isomorphic to trivial bundles, then the only such maps are either Radon transform¹, or the α -cosine transform S_α for some $\alpha \in \mathbb{C}$, or a rank one operator. The last case is degenerate and less interesting. Nevertheless it can be described explicitly as follows. All such operators are obtained by a twist by a character of $GL(V)$ from a single operator mapping smooth measures on $Gr_i(V)$ to smooth functions on $Gr_j(V)$ given by first integrating a measure and then imbedding this number to functions as a constant function. These three cases are not quite independent: for example in Theorem 4.1 of this paper we will show that in the special case when $j = n - i$ the Radon transform from $Gr_i(V)$ to $Gr_{n-i}(V)$ is proportional to the α -cosine transform for $\alpha = -\min\{i, n - i\}$.

The goal of the present paper is to obtain an analytic description of S_α . We use representation theoretical tools of both $GL(n, \mathbb{R})$ and $O(n)$. Let us describe our main results more precisely. To formulate our first main result let us fix again a Euclidean metric on V . Denote $r := \min\{i, n - i\}$. Then $S_\alpha: C^\infty(Gr_r(V)) \rightarrow C^\infty(Gr_r(V))$. In Section 6, Theorem 6.9, we write down explicitly an $O(n)$ -equivariant differential operator on functions on $Gr_r(V)$, denoted by $\hat{\mathcal{D}}_\nu$ for any $\nu \in \mathbb{C}$, which satisfies the following property.

1.3 Theorem. *If $\alpha \notin [-(r + 1), -2] \cap \mathbb{Z}$ then for some constant $c_\alpha \in \mathbb{C}$ one has*

$$S_\alpha = c_\alpha \cdot \hat{\mathcal{D}}_{\frac{\alpha}{2}} \circ S_{\alpha+2}.$$

The formula for $\hat{\mathcal{D}}_{\frac{\alpha}{2}}$ is explicit but somewhat too technical to be presented in the introduction.

1.4 Example. For $r = 1$ the operator $\hat{\mathcal{D}}_{\frac{\alpha}{2}}$ is a linear combination of the Laplace-Beltrami operator on the unit sphere (restricted to even functions on the sphere) and the identity operator (see Example 6.10 in Section 6 below). For $\alpha = -1$ on the sphere this operator was written down explicitly in [14,

¹The non-trivial observation that the Radon transform between Grassmannians can be rewritten in $GL(V)$ -equivariant terms was first made in [9]; see also Section 4 of this paper for the details.

Proposition 2.1]. Notice also that for $r = 1$ the operator S_{-1} is proportional to the Radon transform (this fact is well known and seems to be a folklore).

1.5 Remark. We have the following mutually excluding cases.

- 1) For $Re(\alpha) > -1$ $S_\alpha = T_\alpha$ is given by the explicit formula (1.1).
- 2) If $Re(\alpha) \leq -1$, but $\alpha \notin -\mathbb{N}$, then by repeated use of Theorem 1.3 the operator S_α can be expressed as a composition of some explicit (though complicated) $O(n)$ -equivariant differential operator and S_β with $Re(\beta) > -1$ (thus S_β is given by (1.1)). Clearly $Re(\beta)$ can be chosen to be arbitrarily large.
- 3) If $\alpha \in -r - 2\mathbb{N}$ then S_α is composition of an explicit $O(n)$ -equivariant differential operator with S_{-r} .
- 4) If $\alpha \in -(r+1) - 2\mathbb{N}$ then S_α is a composition of an explicit $O(n)$ -equivariant differential operator with $S_{-(r+1)}$.
- 5) It remains to consider $\alpha = -(r+1), -r, \dots, -1$. (Notice however that by Theorem 1.3 one has $S_{-1} = const \cdot \hat{\mathcal{D}}_{-\frac{1}{2}} \circ S_1$.) Our second main result below is an analytic interpretation of S_α in all these cases except $\alpha = -(r+1)$ for which we do not know a good interpretation.

1.6 Theorem. *Let $r := \min\{i, n-i\}$ as previously. We have*

- 1) $S_{-r}: C^\infty(Gr_i(V)) \rightarrow C^\infty(Gr_{n-i}(V))$ is the Radon transform; it can be rewritten as a $GL(V)$ -equivariant operator (see Section 4 for details; if $i = n/2$ the Radon transform coincides with the identity operator by convention).
 - 2) Let $\alpha = -(r-1), \dots, -2, -1$. Then $S_\alpha: C^\infty(Gr_i(V)) \rightarrow C^\infty(Gr_{n-i}(V))$ can be presented in two different ways:
 - (a) as a composition of two Radon transforms via the Grassmannian $Gr_{|\alpha|}(V)$;
 - (b) as a composition of two Radon transforms via the Grassmannian $Gr_{n-|\alpha|}(V)$.
- In both cases (a) and (b) one can rewrite the spaces and the operators in $GL(V)$ -equivariant language (see Section 5 for details).*

This result is a combination of Theorems 4.1 and 5.1 below.

1.7 Example. Let $r = 1$. This is the case of even functions on the sphere. As we have already mentioned in Example 1.4 above, Theorem 1.6 says that S_{-1} is (up to a constant) the Radon transform on the sphere.

The last main result of this paper is Theorem 1.8 below which computes explicitly one more characteristic of the α -cosine transform S_α : the support of the distributional kernel. We refer for the details to Section 7. Here we remind only that by the Schwartz kernel theorem S_α is given by a kernel on $Gr_i(V) \times Gr_{n-i}(V)$ which is a generalized section of an appropriate line bundle over that space. Since S_α is $GL(V)$ -equivariant operator, this section is $GL(V)$ -invariant. Hence its support is a compact $GL(V)$ -invariant subset of $Gr_i(V) \times Gr_{n-i}(V)$.

To describe this support, let us observe that $GL(V)$ has finitely many orbits on $Gr_i(V) \times Gr_{n-i}(V)$. Each orbit consists of pairs of subspaces (E^i, F^{n-i}) such that the dimension of their intersection is equal to a given number. Thus for each $l = 0, 1, \dots, r$ define the orbit

$$\mathcal{O}_l := \{(E^i, F^{n-i}) \mid \dim(E^i \cap F^{n-i}) = l\}.$$

Clearly $\bar{\mathcal{O}}_l \supset \bar{\mathcal{O}}_k$ iff $l \leq k$.

1.8 Theorem. (a) If $\alpha \neq -1, -2, -3, -4, \dots$ then the support of the distributional kernel is $\bar{\mathcal{O}}_0$, i.e. maximal.

(b) If $\alpha = -1, -2, \dots, -r+1$, then the support is equal $\bar{\mathcal{O}}_{|\alpha|}$.

(c) If $\alpha \in -r - 2\mathbb{Z}_{\geq 0}$ then the support is $\bar{\mathcal{O}}_r$, i.e. minimal.

(d) If $\alpha \in -r - 1 - 2\mathbb{Z}_{\geq 0}$ then the support is $\bar{\mathcal{O}}_{r-1}$ (i.e. next to minimal).

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2 Some linear algebraic isomorphisms.

In this section we describe several canonical isomorphisms from (multi-) linear algebra. All this material is well known and standard, but we would like to review it for the sake of completeness. In the rest of the paper we will use these isomorphisms even without mentioning them explicitly.

Let X be an n -dimensional real vector space. In this section all constructed spaces will be real, but in subsequent sections we will sometimes use the same notation for real spaces (more precisely, real vector bundles) and their complexifications. We denote

$$\det X := \wedge^n X.$$

Let $Y \subset X$ be an m -dimensional linear subspace. Then we have a canonical isomorphism

$$\det Y \otimes \det(X/Y) \xrightarrow{\sim} \det X \quad (2.1)$$

which is given by

$$(v_1 \wedge \cdots \wedge v_l) \otimes (\xi_1 \wedge \cdots \wedge \xi_{n-l}) \mapsto v_1 \wedge \cdots \wedge v_l \wedge \bar{\xi}_1 \wedge \cdots \wedge \bar{\xi}_{n-l},$$

where $\bar{\xi}_i \in V$ is a lift of $\xi_i \in X/Y$. It is easy to see that the map (2.1) is well defined and is an isomorphism.

We have another canonical isomorphism

$$\det X^* \simeq (\det X)^* \quad (2.2)$$

which is induced by the perfect pairing $\det X^* \times \det X \rightarrow \mathbb{R}$ given by

$$(v_1^* \wedge \cdots \wedge v_n^*, w_1 \wedge \cdots \wedge w_n) \mapsto \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n v_i^*(w_{\sigma(i)}),$$

where the sum runs over all the permutations of length n .

Let us define the orientation space $or(X)$ of the vector space X . Let $\mathcal{B}(X)$ denote the set of all basis of X . The group $GL(X)$ acts freely and transitively on $\mathcal{B}(X)$. By definition $or(X)$ is the space of functions $f: \mathcal{B}(X) \rightarrow \mathbb{R}$ such that

$$f(g(e)) = \operatorname{sgn}(\det(g))f(e)$$

for any basis $e \in \mathcal{B}(X)$ and any $g \in GL(X)$. Clearly $or(X)$ is 1-dimensional.

Any basis of X defines a basis of $or(X)$ which is the function from $\mathcal{B}(X)$ equal to 1 on this particular basis. Let us define a canonical isomorphism

$$or(Y) \otimes or(X/Y) \xrightarrow{\sim} or(X). \quad (2.3)$$

Choose a basis $v_1, \dots, v_m \in Y$; let $f_Y \in or(Y)$ be the function from $\mathcal{B}(Y)$ equal to 1 on this basis. Fix a basis $\xi_1, \dots, \xi_{n-m} \in X/Y$; let $f_{X/Y} \in or(X/Y)$ be the function equal to 1 on this basis. Fix a lift $\bar{\xi}_i \in X$ for $i = 1, \dots, n-m$. Then $v_1, \dots, v_m, \bar{\xi}_1, \dots, \bar{\xi}_{n-m}$ is a basis of X . Let f_X be the function equal to 1 on that basis. Then there exists unique isomorphism (2.3) such that

$$f_Y \otimes f_{X/Y} \mapsto f_X.$$

It is not hard to check that this map is independent of the choices of bases.

By taking dual basis one easily constructs canonical isomorphisms

$$or(X^*) \simeq or(X) \simeq or(X)^*. \quad (2.4)$$

Let $D(X)$ denote the space of real valued Lebesgue measures on X (as we mentioned at the beginning of this section, in subsequent sections $D(X)$ will denote the complexification of this space, namely the space of complex valued Lebesgue measures.) $D(X)$ is 1-dimensional. There is a canonical isomorphism

$$\det X^* \otimes or(X) \xrightarrow{\sim} D(X) \quad (2.5)$$

which is described as follows. Fix a basis $e_1, \dots, e_n \in X$. Let $f \in \mathcal{B}(X)$ be the function equal to 1 on this basis. Let $e_1^*, \dots, e_n^* \in X^*$ be the dual basis. Then there exists a unique isomorphism (2.6) such that $(e_1^* \wedge \dots \wedge e_n^*) \otimes f$ is mapped to the Lebesgue measure whose value on the parallelepiped spanned by e_1, \dots, e_n is equal to 1.

Isomorphisms (2.1), (2.2), (2.3), (2.5) imply the canonical isomorphisms

$$D(Y) \otimes D(X/Y) \simeq D(X), \quad (2.6)$$

$$D(X^*) \simeq (D(X))^*. \quad (2.7)$$

One can easily check the following canonical isomorphisms

$$D(X \otimes Y) \simeq D(X)^{\otimes \dim Y} \otimes D(Y)^{\otimes \dim X}, \quad (2.8)$$

$$\det(X \otimes Y) \simeq \det(X)^{\otimes \dim Y} \otimes \det(Y)^{\otimes \dim X}, \quad (2.9)$$

$$or(X \otimes Y) \simeq or(X)^{\otimes \dim Y} \otimes or(Y)^{\otimes \dim X} \quad (2.10)$$

For $\alpha \in \mathbb{C}$ let us define the space $D^\alpha(X)$ of α -densities on a real n -dimensional vector space X as the space of functions $f: \mathcal{B}(X) \rightarrow \mathbb{R}$ satisfying

$$f(g(e)) = |\det g|^\alpha f(e) \text{ for any } g \in GL(X), e \in \mathcal{B}(X).$$

Clearly $\dim D^\alpha(X) = 1$. We have a canonical isomorphism

$$D(X) \xrightarrow{\sim} D^1(X).$$

It is defined as follows: any Lebesgue measure μ on X defines the function on $\mathcal{B}(X)$ whose value at a basis $e \in \mathcal{B}(X)$ is equal to the μ -measure of the parallelepiped spanned by the basis e .

Furthermore we have the following canonical isomorphisms (where $Y \subset X$ is a linear subspace):

$$\begin{aligned} D^\alpha(Y) \otimes D^\alpha(X/Y) &\simeq D^\alpha(X), \\ D^\alpha(X) \otimes D^\beta(X) &\simeq D^{\alpha+\beta}(X), \\ (D^\alpha(X))^* &\simeq D^\alpha(X^*) \simeq D^{-\alpha}(X). \end{aligned}$$

3 Invariant form of the α -cosine transform.

Let us describe an invariant description of the α -cosine transform following [6]. This is going to be a $GL(V)$ -invariant operator

$$T_\alpha: C^\infty(Gr_i(V), L_\alpha) \rightarrow C^\infty(Gr_{n-i}(V), M_\alpha),$$

where L_α, M_α are $GL(V)$ -equivariant line bundles over corresponding Grassmannians, $n = \dim V$. The line bundles are described as follows. For subspaces $E^i \in Gr_i(V)$, $F^{n-i} \in Gr_{n-i}(V)$ the fibers are

$$L_\alpha|_{E^i} = D^\alpha(V/E^i) \otimes |\omega_{Gr_i}|_{|E^i}, \quad M_\alpha|_{F^{n-i}} = D^\alpha(F^{n-i}),$$

where $D^\alpha(X)$ denotes the space of α -densities on a vector space X , and $|\omega_{Gr_i}|$ denotes the line bundle of densities over $Gr_i(V)$, i.e. its fiber over a point p is equal to the space of complex valued Lebesgue measures on the tangent space of p . In this notation we have

$$(T_\alpha f)(F) = \int_{E \in Gr_i(V)} pr_{F \rightarrow V/E}^* (f(E)), \quad (3.1)$$

where $pr_{F \rightarrow V/E}: F \rightarrow V/E$ is the restriction to F of the canonical projection $V \rightarrow V/E$, and the star $*$ denotes the pull-back on α -densities. It is obvious that T_α is well defined for $Re(\alpha) \geq 0$, namely the integral (3.1) converges. Actually by [6], Lemma 1.1, the integral converges absolutely for $Re(\alpha) > -1$. Since T_α coincides with the intertwining integral, by a general result (see e.g. [31, Theorem 10.1.6]) T_α has a meromorphic continuation in $\alpha \in \mathbb{C}$. For any $\alpha_0 \in \mathbb{C}$ we will denote by S_{α_0} the operator which is the leading non-zero coefficient in the decomposition of T_α at α_0 . Thus

$$S_\alpha: C^\infty(Gr_i(V), L_\alpha) \rightarrow C^\infty(Gr_{n-i}(V), M_\alpha) \quad (3.2)$$

is a non-zero $GL(V)$ -equivariant operator.

For future reference, let us rewrite the line bundles L_α, M_α slightly more explicitly. Below for a vector space W we denote $D^{-\gamma}(W)$ by $|\det(W)|^{\otimes \gamma}$. Then we have

$$L_\alpha|_{E^i} = |\det E^i|^{\otimes(n+\alpha)} \otimes |\det V|^{\otimes-(i+\alpha)}, \quad (3.3)$$

$$M_\alpha|_{F^{n-i}} = |\det F^{n-i}|^{\otimes-\alpha}, \quad (3.4)$$

where we have use the well known canonical isomorphism for the tangent space

$$T_{E^i}Gr_i(V) = (E^i)^* \otimes V/E^i.$$

4 Invariant form of the Radon transform.

Let us denote by R_{ji} the Radon transform from the i -Grassmannian to the j -Grassmannian in an n -dimensional vector space V . In order to describe it in $GL(V)$ -invariant terms it is convenient to consider two cases: $i < j$ and $i > j$, while for $i = j$ the Radon transform R_{ii} is the identity operator. First the Radon transform between Grassmannians was rewritten in $GL(V)$ -equivariant term in [9].

Case $i < j$. In this case

$$R_{ji}: C^\infty(Gr_i(V), L) \rightarrow C^\infty(Gr_j(V), M),$$

where

$$\begin{aligned} L|_{E^i} &= |\det E^i|^{\otimes j} \text{ for any } E^i \in Gr_i(V), \\ M|_{F^j} &= |\det F^j|^{\otimes i} \text{ for any } F^j \in Gr_j(V). \end{aligned}$$

Then

$$(R_{ji}f)(F^j) = \int_{E^i \in Gr_i(F^j)} f(E^i),$$

where the last expression makes sense because

$$f(E^i) \in L|_{E^i} = M|_{F^j} \otimes |\omega_{Gr_i(F^j)}|_{E^i},$$

where $|\omega_{Gr_i(F^j)}|$ denotes the line bundle of densities on $Gr_i(F^j)$ so that its sections can be integrated over $Gr_i(F^j)$.

Case $i > j$. In this case

$$R_{ji}: C^\infty(Gr_i(V), S) \rightarrow C^\infty(Gr_j(V), T),$$

where

$$\begin{aligned} S|_{E^i} &= |\det E^i|^{\otimes(n-j)} \otimes |\det V|^{\otimes-(i-j)}, \\ T|_{F^j} &= |\det F^j|^{\otimes(n-i)}. \end{aligned}$$

Moreover for $E^i \supset F^j$

$$S|_{E^i} = T|_{F^j} \otimes |\omega|_{E^i},$$

where $|\omega|$ denotes the line bundle of densities over the manifold of i -dimensional linear subspaces containing the given j -dimensional subspace F^j . Hence the invariant form of R_{ji} is given by

$$(R_{ji}f)(F^j) = \int_{E^i \supset F^j} f(E^i).$$

It was shown in [9] that R_{ji} has maximal possible range, more precisely R_{ji} is onto if $\dim Gr_i(V) \geq \dim Gr_j(V)$, and it is a closed imbedding if $\dim Gr_i(V) \leq \dim Gr_j(V)$. In particular R_{ji} is an isomorphism if $j = n - i$.

4.1 Theorem. (1) Let $i \neq n/2$, $r := \min\{i, n - i\}$. Then the $(-r)$ -cosine transform S_{-r} is proportional to the Radon transform $R_{n-i,i}$.

(2) Let $i = n/2$, thus $r = n/2$. Then the cosine transform S_{-r} is proportional to the identity operator on the space of sections of the line bundle over $Gr_{\frac{n}{2}}(V)$ whose fiber over $E \in Gr_{\frac{n}{2}}(V)$ is equal to $|\det E|^{\otimes \frac{n}{2}}$.

Proof. In the case (1) S_{-r} and $R_{n-i,i}$ act between the same pair of line bundles by the above discussion of this section and by (3.3)-(3.4). Hence by the uniqueness theorem [15], which says that the space of $GL(V)$ -equivariant operators between spaces of sections of $GL(V)$ -equivariant line bundles over two Grassmannians is at most one dimensional², they must be proportional. In the case (2) S_{-r} acts on the space of sections of the line bundle described in the statement of the theorem due to (3.3)-(3.4). Hence by the uniqueness theorem of [15] again S_{-r} must be proportional to the identity operator.

²In this paper the uniqueness theorem [15] is used in a special case of a pair of i - and $(n - i)$ -dimensional Grassmannians. In this case it easily follows from Theorem 7.17 of this paper.

5 Composition of Radon transforms

In either case of the previous section ($i < j$ or $i > j$), if we twist the source and the target spaces by the same character of $GL(V)$ and twist the map R_{ji} accordingly, then the obtained map also will be $GL(V)$ -equivariant, and it is natural to call it also Radon transform. We will denote by R'_{ji} such a twist of the Radon transform R_{ji} without specifying the character. Now we will discuss when two (twisted) Radon transforms are composable as $GL(V)$ -equivariant operators, namely $R'_{kj} \circ R'_{ji}$ is well defined. It is easy to see that necessarily $k = n - i$ and there are two cases: $i < j > n - i$ or $i > j < n - i$. Moreover, using a global twist by a character of $GL(V)$, we may assume that one of the composed twisted Radon transforms is not twisted in fact.

Case 1: $i < j > n - i$. We have a composition of maps

$$C^\infty(Gr_i(V), L) \xrightarrow{R_{ji}} C^\infty(Gr_j(V), M) \xrightarrow{R'_{n-i,j}} C^\infty(Gr_{n-i}(V), T),$$

where

$$L|_{E^i} = |\det E^i|^{\otimes j} \text{ for any } E^i \in Gr_i(V), \quad (5.1)$$

$$M|_{F^j} = |\det F^j|^{\otimes i} \text{ for any } F^j \in Gr_j(V), \quad (5.2)$$

$$T|_{G^{n-i}} = |\det G^{n-i}|^{\otimes(n-j)} \otimes |\det V|^{\otimes(i+j-n)} \text{ for any } G^{n-i} \in Gr_{n-i}(V). \quad (5.3)$$

Case 2: $i > j < n - i$. We have a composition of maps

$$C^\infty(Gr_i(V), L) \xrightarrow{R_{ji}} C^\infty(Gr_j(V), M) \xrightarrow{R'_{n-i,j}} C^\infty(Gr_{n-i}(V), T),$$

where

$$\begin{aligned} L|_{E^i} &= |\det E^i|^{\otimes(n-j)}, \\ M|_{F^j} &= |\det F^j|^{\otimes(n-i)} \otimes |\det V|^{\otimes(i-j)}, \\ T|_{G^{n-i}} &= |\det G^{n-i}|^{\otimes j} \otimes |\det V|^{\otimes(i-j)}. \end{aligned}$$

Now let us observe that $i > j < n - i$ if and only if $i < n - j > n - i$. Hence according to cases 1 and 2, for $j < \min\{i, n - i\}$ we have two compositions of the Radon transforms between the same spaces:

$$R'_{n-i,j} \circ R_{ji}, R'_{n-i,n-j} \circ R_{n-j,i}: C^\infty(Gr_i(V), L) \rightarrow C^\infty(Gr_{n-i}(V), T), \quad (5.4)$$

$$\text{where } L|_{E^i} = |\det E^i|^{\otimes(n-j)}, T|_{G^{n-i}} = |\det G^{n-i}|^{\otimes j} \otimes |\det V|^{\otimes(i-j)}. \quad (5.5)$$

Now let us observe that, due to (3.3)-(3.4) and (5.4-5.5), the $(-j)$ -cosine transform twisted by a character acts between the same spaces:

$$S_{-j} \otimes |\det(\cdot)|^{\otimes(i-j)}: C^\infty(Gr_i(V), L) \rightarrow C^\infty(Gr_{n-i}(V), T).$$

5.1 Theorem. *Let $1 \leq j < \min\{i, n-i\}$. Then the three $GL(V)$ -intertwining operators $R'_{n-i,j} \circ R_{ji}$, $R'_{n-i,n-j} \circ R_{n-j,i}$, and $S_{-j} \otimes |\det(\cdot)|^{\otimes(i-j)}$ between*

$$C^\infty(Gr_i(V), L) \rightarrow C^\infty(Gr_{n-i}(V), T)$$

are proportional to each other.

Proof. This immediately follows from the uniqueness theorem [15]. Q.E.D.

6 The precise statement and proof of Theorem 1.3.

In this section we will use representation theory of $SO(n)$ and $O(n)$ on functions on real Grassmannians. First let us remind few standard representation theoretical facts (see e.g. [33]).

All irreducible (continuous) representations of $SO(n)$ or $O(n)$ are necessarily finite dimensional. For $SO(n)$ they can be parameterized by highest weights. In turn, highest weights of $SO(n)$ can be identified with some combinatorial data, namely with sequences of integers

$$(m_1, \dots, m_{l-1}, m_l) \text{ where } l := \lfloor n/2 \rfloor,$$

which satisfy

$$\begin{aligned} m_1 &\geq \dots m_{l-1} \geq m_l \geq 0 \text{ if } n \text{ is odd,} \\ m_1 &\geq \dots m_{l-1} \geq |m_l| \text{ if } n \text{ is even.} \end{aligned}$$

The natural representation of $SO(n)$ or $O(n)$ on $L^2(Gr_i(\mathbb{R}^n))$ is isomorphic to such a representation on $L^2(Gr_{n-i}(\mathbb{R}^n))$ using the $O(n)$ -equivariant identification $Gr_i(\mathbb{R}^n) \simeq Gr_{n-i}(\mathbb{R}^n)$ by taking the orthogonal complement. Hence denote $r := \min\{i, n-i\}$ and consider the action of $SO(n)$ and $O(n)$ on $L^2(Gr_r(\mathbb{R}^n))$. These are unitary representations and each irreducible representation of either group is known to enter $L^2(Gr_r(\mathbb{R}^n))$ with multiplicity

at most 1. Moreover each $SO(n)$ - or $O(n)$ -irreducible subspace consists of infinitely smooth functions.

It is well known that an irreducible $SO(n)$ -module with the highest weights (m_1, \dots, m_l) does enter the decomposition of $L^2(Gr_r(\mathbb{R}^n))$ if and only if:

- all m_i are even integers;
- $m_i = 0$ for $i > r$.

In addition we will need a description of the decomposition of $L^2(Gr_r(\mathbb{R}^n))$ under the action of $O(n)$. Let us denote for $1 \leq d \leq r \leq n/2$

$$\begin{aligned}\Lambda_r &:= \{(m_1, \dots, m_r) \in (2\mathbb{Z})^r \mid m_1 \geq \dots \geq m_r \geq 0\}, \\ \Lambda_{d,r} &:= \{m \in \Lambda_r \mid m_d = \dots = m_r = 0\}.\end{aligned}$$

Let us consider two cases: $r < n/2$ and $r = n/2$.

Case 1. $r < n/2$.

Each $SO(n)$ -irreducible subspace of $L^2(Gr_r(\mathbb{R}^n))$ is $O(n)$ -invariant (and of course irreducible). Hence we can write

$$L^2(Gr_r(\mathbb{R}^n)) = \oplus_{m \in \Lambda_r} \mathcal{H}_m,$$

where \mathcal{H}_m is an irreducible representation of $O(n)$ such that its restriction to $SO(n)$ is still irreducible and has highest weight $(m_1, \dots, m_r, \underbrace{0, \dots, 0}_{l-r \text{ times}})$.

Case 2. $r = n/2$.

Each $SO(n)$ -irreducible subspace of $L^2(Gr_r(\mathbb{R}^n))$ with highest weight $\bar{m} := (m_1, \dots, m_{r-1}, m_r = 0)$ is $O(n)$ -invariant (and clearly irreducible). We will denote this subspace \mathcal{H}_m .

However if $m_r \neq 0$ the $SO(n)$ -irreducible subspace is *not* $O(n)$ -invariant. What happens it is that the sum of two $SO(n)$ -irreducible subspaces with highest weights $(m_1, \dots, m_{r-1}, m_r)$ and $(m_1, \dots, m_{r-1}, -m_r)$ is $O(n)$ -invariant and $O(n)$ -irreducible subspace. Let us denote this subspace by \mathcal{H}_m where $m := (m_1, \dots, m_{r-1}, |m_r|)$. Thus for $r = n/2$ we have again

$$L^2(Gr_r(\mathbb{R}^n)) = \oplus_{m \in \Lambda_r} \mathcal{H}_m.$$

To summarize, in both cases ($r < n/2$ and $r = n/2$) we obtain that

$$L^2(Gr_r(\mathbb{R}^n)) = \oplus_{m \in \Lambda_r} \mathcal{H}_m,$$

where \mathcal{H}_m are pairwise non-isomorphic irreducible representations of $O(n)$ described above.

This language will be very useful in our paper due to Theorem 6.1 below. To formulate it, let us observe that any (suitably continuous) $O(n)$ -equivariant operator on functions on a Grassmannians $Gr_r(\mathbb{R}^n)$, in particular the α -cosine transforms, leaves each \mathcal{H}_m invariant. By the Schur's lemma it acts on \mathcal{H}_m by multiplication by a scalar.

Let us denote by $c_{\nu,r}(m)$ this scalar for the 2ν -cosine transform on Grassmannian manifold of rank r (i.e. on $Gr_r(V)$ or $Gr_{n-r}(V)$). We denote by $(\nu)_k := \nu(\nu+1)\dots(\nu+k-1)$ for $k \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$ (where for $k=0$ one defines $(\nu)_0 := 1$).

6.1 Theorem ([32], see also [23]).

$$c_{\nu,r}(m) = N_\nu \prod_{j=1}^r \frac{(\nu + \frac{j+1}{2} - \frac{m_j}{2})_{\frac{m_j}{2}}}{(\nu + \frac{n}{2} - \frac{j-1}{2})_{\frac{m_j}{2}}}, \quad (6.1)$$

where N_ν is a meromorphic function of ν which can be written explicitly.

6.2 Remark. Let us observe that zeros of numerators in the expression for $c_{\nu,r}(m)$ are disjoint from zeros of the denominators. Indeed, zeros of the numerator are of the form $-(j-1)/2 + x$, where $j = 1, \dots, r$, and $x \in \mathbb{Z}_{\geq 0}$, while zeros of the denominator are of the form $-n/2 + (k-1)/2 - y$, where again $k = 1, \dots, r$, and $y \in \mathbb{Z}_{\geq 0}$. The smallest zero of the numerator is $-(r-1)/2$, and the largest zero of denominator is $-n/2 + (r-1)/2$. Clearly $-n/2 + (r-1)/2 < -(r-1)/2$.

Let us denote

$$c'_{\nu,r}(m) := \frac{c_{\nu,r}(m)}{N_\nu}.$$

Let $T'_{2\nu} := \frac{1}{N_\nu} T_{2\nu}$; it has the eigenvalues $c'_{\nu,r}(m)$.

6.3 Lemma. (1) $T'_{2\nu}$ depends meromorphically on $\nu \in \mathbb{C}$. It has no zeros.

(2) The poles of $T'_{2\nu}$ are precisely at $\nu \in -\frac{n-j+1}{2} - \mathbb{Z}_{\geq 0}$, where $j = 1, \dots, r$.

(3) The multiplicity $\mu(l)$ of the pole of $T'_{2\nu}$ at $\nu = l \in \frac{1}{2}\mathbb{Z}$ is computed as follows:

(a) if $l > -\frac{n}{2} + \frac{r-1}{2}$ then $\mu(l) = 0$;

(b) if $l \leq -\frac{n}{2}$ then: if $l + n/2 \in \mathbb{Z}$ then $\mu(l) = \lceil \frac{r}{2} \rceil$; and if $l + n/2 \notin \mathbb{Z}$ then $\mu(l) = \lfloor \frac{r}{2} \rfloor$;

(c) if $-\frac{n}{2} < l \leq -\frac{n}{2} + \frac{r-1}{2}$ then $\mu(l) = \lfloor \frac{r-1-n}{2} - l \rfloor + 1$.

Proof. (1), (2) follow from Remark 6.2; the absence of zeros follows from the observation that $c'_{\nu,r}(0) = 1$. Part (3) follows by counting zeros of denominators in the right hand side of (6.1). Q.E.D.

Let us define

$$d_{\nu,r}(m) = \frac{c'_{\nu,r}(m)}{c'_{\nu+1,r}(m)} = \prod_{j=1}^r \frac{(\nu + \frac{j+1}{2} - \frac{m_j}{2}) \cdot (\nu + \frac{n}{2} - \frac{j-1}{2} + \frac{m_j}{2})}{(\nu + \frac{j+1}{2}) \cdot (\nu + \frac{n}{2} - \frac{j-1}{2})}. \quad (6.2)$$

Let us denote by \mathcal{D}_ν the operator with eigenvalues $d_{\nu,r}(m)$. We will see below that \mathcal{D}_ν is a differential operator with coefficients depending meromorphically on ν . Clearly

$$T'_{2\nu} = \mathcal{D}_\nu \circ T'_{2\nu+2}. \quad (6.3)$$

Clearly (6.2) immediately implies the following lemma.

6.4 Lemma. *The operator \mathcal{D}_ν does not vanish identically for any ν , and the poles are precisely at $\nu = -\frac{j+1}{2}$ and $-\frac{n}{2} + \frac{j-1}{2}$ with $j = 1, \dots, r$ (notice that if $r = n/2$ then $-\frac{r+1}{2}$ appears twice in the list). Moreover if $r \neq n/2$ then all the poles are simple; if $r = n/2$ then the pole at $\nu = -\frac{r+1}{2}$ has multiplicity 2, while all the other poles are simple.*

Clearly for any $\nu_0 \in \mathbb{C}$ the operator $S_{2\nu_0}$ is equal to

$$S_{2\nu_0} = \lim_{\nu \rightarrow \nu_0} (\nu - \nu_0)^\mu \cdot T'_{2\nu},$$

where μ is the multiplicity of the pole of $T'_{2\nu}$ at ν_0 .

Similarly we denote the operator

$$\tilde{\mathcal{D}}_{\nu_0} := \lim_{\nu \rightarrow \nu_0} (\nu - \nu_0)^\tau \mathcal{D}_\nu,$$

where τ is the multiplicity of the pole of \mathcal{D}_ν at ν_0 .

Let us also denote by $\hat{\mathcal{D}}_\nu$ the operator

$$\hat{\mathcal{D}}_\nu := \left(\prod_{j=1}^r (\nu + \frac{j+1}{2}) \cdot (\nu + \frac{n}{2} - \frac{j-1}{2}) \right) \cdot \mathcal{D}_\nu.$$

From (6.2) it is clear that for any $\nu \in \mathbb{C}$ the operator $\hat{\mathcal{D}}_\nu$ acts on the irreducible $O(n)$ -representation \mathcal{H}_m as multiplication by a scalar

$$\prod_{j=1}^r \left(\nu + \frac{j+1}{2} - \frac{m_j}{2} \right) \cdot \left(\nu + \frac{n}{2} - \frac{j-1}{2} + \frac{m_j}{2} \right).$$

Thus it is clear that for any ν the operator $\hat{\mathcal{D}}_\nu$ does not vanish identically, and $\hat{\mathcal{D}}_\nu$ and $\tilde{\mathcal{D}}_\nu$ are proportional to each other. Below we will see, following [18], that $\hat{\mathcal{D}}_\nu$ is an $O(n)$ -invariant differential operator on functions on the Grassmannian which will be written down more explicitly.

Our next goal is the following result.

6.5 Proposition. *Let $l \notin [-\frac{r+1}{2}, \dots, -1] \cap \frac{1}{2}\mathbb{Z}$. Then there exists a constant $c_l \in \mathbb{C}$ such that*

$$S_{2l} = c_l \cdot \hat{\mathcal{D}}_l \circ S_{2l+2}.$$

Proof. The proof is by a careful investigation of the orders of the poles in (6.3). Below we will denote by $\mu(T'_{2l})$ (resp. $\mu(\mathcal{D}_l)$) the order of the pole of T'_{2l} (resp. \mathcal{D}_l) at l .

Since the poles ν of $T'_{2\nu}$ and \mathcal{D}_ν are half-integers, the result obviously holds for l not a half-integer. Thus below we will always assume that $l \in \frac{1}{2}\mathbb{Z}$ and l does not belong to the segment $[-\frac{r+1}{2}, \dots, -1]$.

Case 1: $l > -1$. In this case $\mu(T'_{2l}) = \mu(T'_{2l+2}) = \mu(\mathcal{D}_l) = 0$. Hence in this case the result follows immediately from (6.3).

Case 2: $l \leq -\frac{n}{2} - 1$. In this case $\mu(T'_{2l}) = \mu(T'_{2l+2}) =: \mu$ and both are equal either to $\lceil r/2 \rceil$ or $\lfloor r/2 \rfloor$ by Lemma 6.3(3)(b). Thus $\mu(\mathcal{D}_l) = 0$. Multiplying both sides of (6.3) by $(\nu - l)^\mu$ we conclude the proposition is this case.

Case 3: $l = -\frac{n}{2} - \frac{1}{2}$. We have by Lemma 6.3

$$\mu(T'_{2l}) = \mu(T'_{2l+2}) = \lfloor r/2 \rfloor, \mu(\mathcal{D}_l) = 0.$$

Multiplying (6.3) by $(\nu - l)^{\lfloor r/2 \rfloor}$ we deduce the result.

Case 4: $l = -\frac{n}{2}$. We have

$$\mu(\mathcal{D}_l) = 1, \mu(T'_{2l}) = \lceil r/2 \rceil. \tag{6.4}$$

Subcase 4a: assume $r \geq 3$. Then $l + 1 \in (-\frac{n}{2}, -\frac{n}{2} + \frac{r-1}{2}]$. Hence

$$\mu(T'_{2l+2}) = \lfloor \frac{r-1}{2} \rfloor.$$

It is easy to see that $\lceil r/2 \rceil = \lfloor \frac{r-1}{2} \rfloor + 1$. Hence $\mu(T'_{2l}) = \mu(T'_{2l+2}) + 1$. Hence multiplying (6.3) by $(\nu - l)^{\mu(T'_{2l})}$ we get the proposition in subcase 4a.

Subcase 4b: assume $r = 1, 2$. By Lemma 6.3(3)(b)

$$\mu(T'_{2l+2}) = 0.$$

But by (6.4)

$$\mu(T'_{2l}) = \mu(\mathcal{D}_l) = 1.$$

Multiplying (6.3) by $\nu - l$ we get the result in the subcase 4b. Thus case 4 is proven.

Case 5: $-\frac{n}{2} < l \leq -\frac{n}{2} + \frac{r-1}{2}$. First recall that from the very beginning we have assumed that $l \notin [-\frac{r+1}{2}, \dots, -1]$. Hence if $l = -\frac{n}{2} + \frac{r-1}{2}$, in that case it follows that $r < n/2$. Lemma 6.3(3) implies that

$$\mu(T'_{2l}) = \mu(T'_{2l+2}) + 1.$$

Also $\mu(\mathcal{D}_l) = 1$ (here we have used $r < n/2$ if $l = -\frac{n}{2} + \frac{r-1}{2}$; otherwise the order of the pole would be equal to 2). Hence multiplying (6.3) by $(\nu - l)^{\mu(T'_{2l})}$ we get the proposition in case 5.

Case 6: $-\frac{n}{2} + \frac{r-1}{2} < l < -\frac{r+1}{2}$. This is the last case to be checked. In this case we necessarily have $r < n/2$. We have

$$\mu(\mathcal{D}_l) = 0.$$

Also by Lemma 6.3(3)(a)

$$\mu(T'_{2l}) = \mu(T'_{2l+2}) = 0.$$

Hence the result follows. Q.E.D.

Now we are going to describe $\hat{\mathcal{D}}_\nu$ as a differential operator. Recall that the Pfaffian of a $2k \times 2k$ skew-symmetric matrix $M = (M_{ij})$ is

$$Pf(M) = \frac{1}{k!} \sum_{\sigma} \text{sgn}(\sigma) M_{\sigma(1)\sigma(2)} \cdots M_{\sigma(2k-1)\sigma(2k)} \quad (6.5)$$

where the sum is over all permutations σ satisfying $\sigma(1) < \sigma(2), \dots, \sigma(2k-1) < \sigma(2k)$. We note that this definition makes sense even if the entries of M belong to a non-commutative algebra, such as an enveloping algebra.

Now let X be the $n \times n$ skew-symmetric matrix whose ij -th entry is

$$X_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}(n),$$

where E_{ij} is the $n \times n$ matrix all whose entries vanish but the entry on i th row and j th column is equal to 1.

For $I \subset \{1, \dots, n\}$ let X_I be the principal I -minor of X ; if $|I|$ is even then the Pfaffian $Pf(X_I)$, defined by (6.5), is an element of the enveloping algebra $U = U(\mathfrak{o}(n))$. Observe that in the sum (6.5) for $Pf(X_I)$ in each summand all terms commute with each other, while terms from different summands may not commute.

For $d \leq n/2$ we define

$$V_d = (-1)^d \sum_{|I|=2d} Pf(X_I)^2 \in U(\mathfrak{o}(n)). \quad (6.6)$$

6.6 Theorem ([19]). V_d belongs to $(U(\mathfrak{o}(n)))^{O(n)}$.

6.7 Example. (1) $V_0 = 1 \in U(\mathfrak{o}(n))$.

(2) V_1 is proportional to the Casimir element of $\mathfrak{o}(n)$.

6.8 Remark. It was shown in [10, Theorem 2.3] that the algebra $(U(\mathfrak{o}(n)))^{O(n)}$ is a polynomial algebra on (independent) generators $V_1, V_2, \dots, V_{[n/2]}$. However we will not use this fact in the paper.

6.9 Theorem. We have $\hat{D}_\nu = (-\frac{1}{4})^r \sum_{k=0}^r c_k V_k$ where

$$c_k = \prod_{j=k+1}^r [j + 2\nu + 1][j - (2\nu + n + 1)].$$

Notice that in the last theorem $V_k \in (U(\mathfrak{o}(n)))^{O(n)}$ are identified with the differential operators they induce on the Grassmannian $Gr_r(\mathbb{R}^n)$.

6.10 Example. Clearly V_0 induces the identity operator on functions on all Grassmannians. Since $V_1 \in U(\mathfrak{o}(n))$ is proportional to the Casimir element, it induces on functions on each Grassmannian an operator proportional to the Laplace-Beltrami operator. This if $r = 1$ then \hat{D}_ν is a linear combination of the Laplace-Beltrami and the identity operators.

To prove Theorem 6.9 we will need some preparations.

6.11 Lemma. *For $1 \leq d \leq r \leq n/2$ the action of the element V_d vanishes on irreducible $O(n)$ -modules \mathcal{H}_m with $m \in \Lambda_{d,r}$.*

Proof. The irreducible representations of $O(n)$ from $\Lambda_{d,r}$ are precisely those appearing in $L^2(Gr_{d-1}(\mathbb{R}^n))$; each has a vector invariant under $\mathfrak{m} := \mathfrak{o}(d-1) \times \mathfrak{o}(n-d+1)$ (the so called spherical vector). Let us show that each summand in the definition of $Pf(X_I)$, $|I| = 2d$, (see (6.5)) has at least one factor from \mathfrak{m} ; this obviously will imply the lemma.

Let us assume the contrary. Then there exists a subset

$$I = \{i_1 < i_2 < \dots < i_{2d}\} \subset \{1, \dots, n\}$$

and a permutation σ of length $2d$ such that $X_{i_{\sigma(1)}i_{\sigma(2)}}, \dots, X_{i_{\sigma(2d-1)}i_{\sigma(2d)}} \notin \mathfrak{m}$. This implies that for the odd indices

$$1 \leq i_{\sigma(1)}, i_{\sigma(3)}, \dots, i_{\sigma(2d-1)} \leq d-1.$$

Since all these d numbers are distinct, we get a contradiction. Q.E.D.

Let $1 \leq r \leq n/2$. Let $\mathbb{D} = \mathbb{D}(Gr_r(\mathbb{R}^n))$ be the algebra of $O(n)$ -invariant differential operators on $Gr_r(\mathbb{R}^n)$, and let \mathbb{D}_k be the subspace of operators of order $\leq k$. Also let $\mathbb{P} = \mathbb{P}(z_1, \dots, z_r)$ be the algebra of polynomials in (z_1, \dots, z_r) , which are invariant under sign changes and permutations of coordinates, namely under all transformations of the form

$$(z_1, \dots, z_r) \mapsto (\pm z_{\sigma(1)}, \dots, \pm z_{\sigma(r)})$$

with σ being any permutation of length r . Let \mathbb{P}_k be the subspace of polynomials with degree $\leq k$. Harish-Chandra's theorem for the symmetric space $Gr_r(\mathbb{R}^n)$ can be formulated as follows.

6.12 Theorem. *There is a unique algebra isomorphism $\gamma : \mathbb{D} \xrightarrow{\sim} \mathbb{P}$, such that it maps \mathbb{D}_k onto \mathbb{P}_k , and $D \in \mathbb{D}$ acts on the irreducible $O(n)$ -module \mathcal{H}_μ by the scalar $\gamma(D)(\tilde{\mu})$, where*

$$\tilde{\mu} := \mu + \rho, \rho := (\rho_1, \dots, \rho_r), \rho_j := n/2 - j.$$

We failed to find a proof of this theorem in literature; its proof will be given in the appendix below.

6.13 Lemma. *Let $1 \leq d \leq r \leq n/2$. Let $\rho = (\rho_1, \dots, \rho_r) \in \mathbb{C}^r$ be an arbitrary vector. The space*

$$\mathcal{V}_d := \{p \in \mathbb{P}_{2d} \mid p(\mu + \rho) = 0 \text{ for all } \mu \in \Lambda_{d,r}\}$$

is at most one dimensional.

Proof. Fix $d \geq 1$ and prove the lemma by the induction in $r \geq d$.

Step 1. Assume $r = d$.

Case 1. Assume $\rho_d = 0$. Then $p(\mu_1, \dots, \mu_{d-1}, 0) = 0$ for any $\mu_1, \dots, \mu_{d-1} \in \mathbb{C}$. Hence p is divisible by z_d . Since p is invariant under sign changes, it is divisible by z_d^2 . Since p is invariant under permutations, it is divisible by $z_1^2 \dots z_d^2$. But $\deg p \leq 2d$. Hence p is proportional to $z_1^2 \dots z_d^2$, and the lemma follows in this case.

Case 2. Assume $\rho_d \neq 0$. Then $p(\mu_1, \dots, \mu_{d-1}, \rho_d) = 0$ for any $\mu_1, \dots, \mu_{d-1} \in \mathbb{C}$. Hence p is divisible by $z_d - \rho_d$ for any $\mu_1, \dots, \mu_{d-1} \in \mathbb{C}$. But since p is invariant under sign changes and all permutations, it is divisible by $\prod_{i=1}^r (z_i^2 - \rho_i^2)$. Since $\deg p \leq 2d$, p is proportional to the latter polynomial.

Step 2. Assume now $r > d$. Let q be another polynomial in z_1, \dots, z_r satisfying the same assumptions of the lemma as p . We have to show that p and q are proportional.

Define the polynomial

$$\hat{p}(z_1, \dots, z_{r-1}) := p(z_1, \dots, z_{r-1}, \rho_r).$$

Clearly \hat{p} is invariant under all permutations and sign changes and $\deg \hat{p} \leq 2d$. Moreover $\hat{p}(\hat{\mu} + \hat{\rho}) = 0$ for any $\hat{\mu} \in \Lambda_{d,r-1}$, where $\hat{\rho} = (\rho_1, \dots, \rho_{r-1})$. Similarly define \hat{q} for q . By the induction assumption \hat{p} and \hat{q} are proportional, say $\hat{p} = \alpha \hat{q}$. Define

$$\tau = p - \alpha q.$$

We are going to show that $\tau \equiv 0$; this will imply the lemma. Clearly τ is a polynomial in z_1, \dots, z_r of degree at most $2d$, invariant under permutations and sign changes. Moreover

$$\hat{\tau}(z_1, \dots, z_{r-1}) := \tau(z_1, \dots, z_{r-1}, \rho_r)$$

vanishes identically. As in Step 1 it follows that τ is divisible by $\prod_{i=1}^r (z_i^2 - \rho_i^2)$ (no matter if ρ_r vanishes or not). But since $\deg \tau \leq 2d < 2r$, it follows that $\tau \equiv 0$, i.e. $p = \alpha q$. Q.E.D.

Let us introduce more notation. Let $x := (x_1, \dots, x_r)$. Let $e_k(x)$ and $h_k(x)$ be the elementary and complete symmetric polynomials in x ,

$$e_k(x) := \sum_{j_1 < \dots < j_k} x_{j_1} \cdots x_{j_k}, \quad h_k(x) := \sum_{j_1 \leq \dots \leq j_k} x_{j_1} \cdots x_{j_k},$$

Then we have

$$\prod_{i=1}^r (1 + tx_i) = \sum_{k=0}^r e_k(x) t^k \quad \prod_{i=1}^r (1 + tx_i)^{-1} = \sum_{k=0}^{\infty} (-1)^k h_k(x) t^k$$

6.14 Lemma. *Let $e_{ij}(x) = e_i(x_j, \dots, x_r)$ and $h_{ij}(x) = h_i(x_j, \dots, x_r)$ then we have*

$$\sum_{j=i}^k (-1)^{k-j} h_{k-j,k} e_{j-i,i+1} = \delta_{ik}.$$

Proof. If $k = i$ then both sides are 1, while if $k < i$ then both sides are 0. Finally, if $k > i$, then the left side is the coefficient c_{k-i} of t^{k-i} in

$$\prod_{l=k}^r (1 + tx_l)^{-1} \prod_{l=i+1}^r (1 + tx_l) = \prod_{l=i+1}^{k-1} (1 + tx_l).$$

But this is a polynomial of degree $k - i - 1$ in t , and hence $c_{k-i} = 0$. Q.E.D.

6.15 Definition. For $0 \leq i, j \leq r$ we put $a_{ij} = b_{ij} = 0$ if $i > j$, and

$$a_{ij} = e_{j-i,i+1}(x), \quad b_{ij} = (-1)^{j-i} h_{j-i,j}(x) \quad \text{if } i \leq j.$$

6.16 Corollary. *The matrices (a_{ij}) and (b_{ij}) are mutual inverses.*

Proof. Lemma 6.14 shows that $\sum_{j=0}^r a_{ij} b_{jk} = \delta_{ik}$, as desired. Q.E.D.

Below we will always assume that $1 \leq d \leq r \leq n/2$ and

$$\rho_j = n/2 - j, \quad j = 1, \dots, r.$$

Now the operator V_d from (6.6) and the isomorphism γ from Theorem 6.12 satisfy

6.17 Theorem. *We have*

$$\gamma(V_d) = \sum_{k=0}^d (-1)^{d-k} h_{d-k}(\rho_d^2, \dots, \rho_r^2) e_k(z_1^2, \dots, z_r^2).$$

Proof. Notice that for $d < n/2$ this is precisely Theorem 9.1(A) in [20]. We will prove the theorem for $d = n/2$ (hence $r = n/2$) though this assumption will be used only in the last step in the computation of the constant of proportionality. We note that the expression on the right is the coefficient v_d of t^d in

$$\prod_{l=d}^r (1 + t\rho_l^2)^{-1} \prod_{l=1}^r (1 + tz_l^2). \quad (6.7)$$

We claim that $\gamma(V_d)$ and v_d both belong to the one-dimensional space \mathcal{V}_d . For $\gamma(V_d)$ this follows by Lemma 6.11, while for v_d we note that if $z \in \{\mu + \rho : \mu \in \Lambda_{d,r}\}$ then

$$z_d = \rho_d, \dots, z_r = \rho_r$$

Thus (6.7) is a polynomial of degree $< d$ in t and hence $v_d(z) = 0$.

By Lemma 6.13 $\gamma(V_d)$ and v_d are proportional. It remains to show that the constant of proportionality is 1. Now we are going to use the assumption $d = n/2$. It suffices to show that for some irreducible $O(n)$ -module \mathcal{H}_μ one has $\gamma(V_d)(\mu + \rho) = v_d(\mu + \rho) \neq 0$. Under the assumption $d = n/2$ we have

$$V_d = (-1)^d Pf(X)^2,$$

where $Pf(X) = \frac{1}{d!} \sum_{\sigma}' sgn(\sigma) X_{\sigma_1 \sigma_2} \cdots X_{\sigma_{n-1} \sigma_n}$ and the sum \sum_{σ}' runs over all permutations σ of length n such that $\sigma_{2i-1} < \sigma_{2i}$ for all i . Since by assumption $d = r = n/2$, then $\rho_d = 0$, and the statement of the theorem reduces to equality

$$\gamma(V_d) = z_1^2 \cdots z_d^2. \quad (6.8)$$

We already know that the two polynomials are proportional, and we have to show that the constant of proportionality is equal to 1. In order to prove that we will compute the action of V_d on $\wedge^d \mathbb{C}^{2d}$, where \mathbb{C}^{2d} is the standard representation of $O(2d)$.

$\wedge^d \mathbb{C}^{2d}$ is an irreducible $O(2d)$ -module (see e.g. [8], Exercise 19.3). However as an $SO(2d)$ -module it is a sum of two irreducible $SO(2d)$ -modules with highest weights $\mu_+ = (\underbrace{1, \dots, 1}_d, 1)$, $\mu_- = (\underbrace{1, \dots, 1}_d, -1)$ (see [8], Theorem 19.2 and Remark (ii) on p. 289 there). Adding ρ to μ_{\pm} we get respectively in these cases

$$z_1 = d, z_2 = d - 1, \dots, z_d = \pm 1.$$

Hence

$$z_1^2 z_2^2 \dots z_d^2 = (d!)^2. \quad (6.9)$$

We are going to show that V_d acts on $\wedge^d \mathbb{C}^{2d}$ as $(d!)^2$; this will finish the proof of the theorem. Let e_1, \dots, e_{2d} be an orthonormal basis of \mathbb{C}^{2d} . We have

$$Pf(X)e_1 \wedge \dots \wedge e_d = \quad (6.10)$$

$$\frac{1}{d!} \sum_{\sigma} sgn(\sigma) \underbrace{(E_{\sigma_1 \sigma_2} - E_{\sigma_2 \sigma_1})}_{X_{\sigma_1 \sigma_2}} \dots \underbrace{(E_{\sigma_{n-1} \sigma_n} - E_{\sigma_n \sigma_{n-1}})}_{X_{\sigma_{n-1} \sigma_n}} e_1 \wedge \dots \wedge e_d. \quad (6.11)$$

Clearly

$$E_{ij}e_p = \begin{cases} e_i, & j = p \\ 0, & j \neq p \end{cases}$$

Hence for $i \neq j$

$$X_{ij}e_p = \begin{cases} e_i, & j = p \\ -e_j, & i = p \\ 0, & i, j \neq p \end{cases}$$

This implies that

$$X_{\sigma_{2i-1} \sigma_{2i}} X_{\sigma_{2l-1} \sigma_{2l}} e_p = 0 \text{ for } i \neq l$$

since $\{\sigma_{2i-1}, \sigma_{2i}\} \cap \{\sigma_{2l-1}, \sigma_{2l}\} = \emptyset$.

Moreover if for some $1 \leq p \leq d$ one has

$$X_{\sigma_{2i-1} \sigma_{2i}} e_p = \pm e_q \text{ with } 1 \leq q \leq d,$$

then such a summand may be omitted from the sum (6.11) since in this case $\{\sigma_{2i-1}, \sigma_{2i}\} = \{p, q\}$, and hence there is no another $X_{\sigma_{2l-1} \sigma_{2l}}$ to apply on e_q .

It follows that the only non-zero summands in (6.11) correspond to $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ of the form:

- $\sigma_1, \sigma_3, \dots, \sigma_{n-1}$ is a permutation of $\{1, \dots, d\}$;
- $\sigma_2, \sigma_4, \dots, \sigma_n$ is a permutation of $\{d+1, \dots, n\}$.

Then

$$(6.11) = (-1)^d \sum_{\tau} sgn(1\tau_1 2\tau_2 \dots d\tau_d) e_{\tau_1} \wedge \dots \wedge e_{\tau_d},$$

where the sum runs over all permutations $\tau = (\tau_1, \dots, \tau_d)$ of the set $\{d+1, \dots, n\}$.

Observe that

$$\text{sgn}(1\tau_1 2\tau_2 \dots d\tau_d) = (-1)^{\frac{d(d-1)}{2}} \text{sgn}(\tau).$$

Hence

$$Pf(X)e_1 \wedge \dots \wedge e_d = (-1)^{d+\frac{d(d-1)}{2}} (d!)e_{d+1} \wedge \dots \wedge e_n. \quad (6.12)$$

Now let us compute $Pf(X)e_{d+1} \wedge \dots \wedge e_n$. Since $\wedge^d \mathbb{C}^{2d}$ is a multiplicity free $SO(2d)$ -module and $Pf(X)$ and the Hodge star \star commute with $SO(2d)$,³ it follows that $Pf(X)$ and \star commute with each other. Then we have

$$\begin{aligned} Pf(X)e_{d+1} \wedge \dots \wedge e_n &= Pf(X)(\star(e_1 \wedge \dots \wedge e_d)) = \star Pf(X)(e_1 \wedge \dots \wedge e_d) \stackrel{(6.12)}{=} \\ \star((-1)^{d+\frac{d(d-1)}{2}} (d!)e_{d+1} \wedge \dots \wedge e_n) &= (-1)^{d+\frac{d(d-1)}{2}} (d!) \cdot (-1)^d e_1 \wedge \dots \wedge e_d. \end{aligned} \quad (6.13)$$

Hence by (6.12) and (6.13) we get

$$\begin{aligned} V_d(e_1 \wedge \dots \wedge e_d) &= (-1)^d Pf(X)^2 e_1 \wedge \dots \wedge e_d = \\ &= (d!)^2 e_1 \wedge \dots \wedge e_d \stackrel{(6.9)}{=} \\ &= z_1^2 \dots z_d^2 \cdot e_1 \wedge \dots \wedge e_d. \end{aligned}$$

The theorem is proved. Q.E.D.

Let us write $\rho^2 = (\rho_1^2, \dots, \rho_r^2)$, then by Definition 6.15, we have

$$a_{ij}(\rho^2) = e_{j-i}(\rho_{i+1}^2, \dots, \rho_r^2), b_{ij}(\rho^2) = (-1)^{j-i} h_{j-i}(\rho_j^2, \dots, \rho_r^2), \text{ for } i \leq j.$$

and so by the previous theorem, the eigenvalues of V_d are

$$\sum_{k=0}^d b_{kd}(\rho^2) e_k(\tilde{\mu}_1^2, \dots, \tilde{\mu}_r^2)$$

By Corollary 6.16, we deduce the following result.

³The fact that $Pf(X)$ commutes with $SO(2d)$ was proven in Proposition 3.6 in [19].

6.18 Theorem. *The operator $E_d = \sum_{k=0}^d a_{kd}(\rho^2) V_k$ has eigenvalues $e_d(\tilde{\mu}_1^2, \dots, \tilde{\mu}_r^2)$ on \mathcal{H}_μ .*

Proof of Theorem 6.9. Recall that the eigenvalue of operator $\hat{\mathcal{D}}_\nu$ on \mathcal{H}_μ is equal to

$$\begin{aligned} & \prod_{j=1}^r \left[\nu + \frac{j+1}{2} - \frac{\mu_j}{2} \right] \left[\nu + \frac{n}{2} - \frac{j+1}{2} + \frac{\mu_j}{2} \right] \\ &= \frac{1}{4^r} \prod_{j=1}^r \left[\left(2\nu + \frac{n}{2} + 1 \right)^2 - \tilde{\mu}_j^2 \right] = \left(\frac{-1}{4} \right)^r \prod_{j=1}^r (\tilde{\mu}_j^2 + \lambda) \end{aligned}$$

where $\lambda := -(2\nu + n/2 + 1)^2$. By Theorem 6.18 we have

$$\hat{\mathcal{D}}_\nu = \left(-\frac{1}{4} \right)^r \sum_{d=0}^r \lambda^{r-d} E_d = \left(-\frac{1}{4} \right)^r \sum_{k=0}^r c_k V_k,$$

where

$$\begin{aligned} c_k &= \sum_{d=k}^r \lambda^{r-d} a_{kd}(\rho^2) = \sum_{d=k}^r \lambda^{r-d} e_{d-k}(\rho_{k+1}^2, \dots, \rho_r^2) = \prod_{j=k+1}^r (\rho_j^2 + \lambda) = \\ & \prod_{j=k+1}^r [j + 2\nu + 1][j - (2\nu + n + 1)]. \end{aligned}$$

Q.E.D.

7 Support of the distributional kernel of the α -cosine transform.

The main goal of this section is to prove Theorem 1.8 of the introduction. Let us fix an isomorphism $V \simeq \mathbb{R}^n$. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Let us consider open subset $\mathcal{U} \subset Gr_{n-r}(\mathbb{R}^n)$ consisting of subspaces E^{n-r} trivially intersecting the r -subspace $\text{span}\{e_{n-r+1}, \dots, e_n\}$. Equivalently E^{n-r} is equal to the span of columns of a matrix $\tilde{X}_{n \times (n-r)}$ of the form

$$\left[\frac{I_{n-r}}{X} \right], \quad (7.1)$$

where X is an $r \times (n - r)$ matrix. Furthermore the map $E^{n-r} \mapsto X$ is a diffeomorphism $\mathcal{U} \xrightarrow{\sim} \text{Mat}_{r \times (n-r)}(\mathbb{R})$.

Let $\mathcal{L}^\beta \rightarrow \text{Gr}_{n-r}(\mathbb{R}^n)$ be the line bundle whose fiber over E^{n-r} is equal to β -densities in E^* :

$$\mathcal{L}^\beta|_{E^{n-r}} = |\det E|^\beta.$$

Let us fix a trivialization of \mathcal{L}^β over \mathcal{U} as follows. Let $E^{n-r} \in \mathcal{U}$ be an arbitrary subspace. Let $X \in \text{Mat}_{r \times (n-r)}(\mathbb{R})$ be the matrix corresponding to E^{n-r} . Thus E^{n-r} is the span of columns of the matrix \tilde{X} defined by (7.1). Let us denote the columns of \tilde{X} by $e_1(E), \dots, e_{n-r}(E)$. Then $e(E)^\beta := |e_1(E) \wedge \dots \wedge e_{n-r}(E)|^{\otimes \beta} \in \mathcal{L}^\beta|_{E^{n-r}}$ defines the required trivialization of \mathcal{L}^β over \mathcal{U} .

Let $Q \subset GL_n(\mathbb{R})$ be the subgroup consisting of block diagonal matrices of the form $\begin{bmatrix} A_{(n-r) \times (n-r)} & 0 \\ 0 & B_{r \times r} \end{bmatrix}$.

We are going to show that \mathcal{U} is Q -invariant under restriction of the natural action of $GL_n(\mathbb{R})$ on $\text{Gr}_{n-r}(\mathbb{R})$ to \mathcal{U} , and compute the action of Q on \mathcal{L}^β .

We have for any $g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in Q$:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_{n-r} \\ X_{r \times (n-r)} \end{bmatrix} = \begin{bmatrix} I_{n-r} \\ BXA^{-1} \end{bmatrix} \cdot A.$$

From this equality we immediately conclude.

7.1 Claim. 1) g maps subspace $E \in \mathcal{U}$ corresponding to a matrix X to subspace $E' \in \mathcal{U}$ corresponding to the matrix BXA^{-1} . In particular \mathcal{U} is Q -invariant.

2) g maps $e(E)^\beta$ to $|\det A|^\beta \cdot e(E')^\beta$.

Let us denote by \mathcal{U}_0 the (closed) submanifold of \mathcal{U} consisting of subspaces containing $E_0^r := \text{span}\{e_1, \dots, e_r\}$. Clearly \mathcal{U}_0 is identified with matrices of the form

$$\tilde{X} = \left[\begin{array}{c|c} I_{n-r} & \\ \hline 0_{r \times r} & Z \end{array} \right],$$

where Z is a $(n - 2r) \times r$ matrix. Let us fix the transversal $\mathcal{N} \subset \mathcal{U}$ to \mathcal{U}_0 :

$$\mathcal{N} = \left\{ \left[\begin{array}{c|c} I_{n-r} & \\ \hline Y_{r \times r} & 0 \end{array} \right] \mid Y \in \text{Mat}_{r \times r}(\mathbb{R}) \right\} \simeq \text{Mat}_{r \times r}(\mathbb{R}), \quad (7.2)$$

where 0 has size $(n - 2r) \times r$.

Let us define the subgroup $Q_0 \subset Q$ consisting of block-diagonal matrices of the form

$$\begin{bmatrix} A_{r \times r} & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & D_{r \times r} \end{bmatrix}.$$

The group Q_0 leaves \mathcal{N} invariant, and we have

$$\begin{bmatrix} A_{r \times r} & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & D_{r \times r} \end{bmatrix} \left[\frac{I_{n-r}}{Y_{r \times r} \mid 0} \right] = \left[\frac{I_{n-r}}{DYA^{-1} \mid 0} \right] \left[\frac{A \mid 0}{0 \mid I_{n-2r}} \right].$$

From the last equality we immediately conclude.

7.2 Claim. *Under the identification $\mathcal{N} \simeq \text{Mat}_{r \times r}(\mathbb{R})$ as in (7.2), an element*

$g = \begin{bmatrix} A_{r \times r} & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & 0 & D_{r \times r} \end{bmatrix} \in Q_0$ *maps a subspace $E \in \mathcal{N}$ corresponding to Y to the subspace $E' \in \mathcal{N}$ corresponding to DYA^{-1} , and maps $e(E)^\beta$ to $|\det A|^\beta \cdot e(E')^\beta$.*

Now let us return back to the α -cosine transform. First consider the case $i \leq n - i$, thus $r = i$. Recall that we have the α -cosine transform

$$S_\alpha: C^\infty(Gr_r(\mathbb{R}^n), L_\alpha) \rightarrow C^\infty(Gr_{n-r}(\mathbb{R}^n), M_\alpha),$$

where

$$\begin{aligned} L_\alpha|_{E^r} &= |\det E|^{\otimes(n+\alpha)} \otimes |\det V|^{\otimes-(r+\alpha)}, \\ M_\alpha|_{F^{n-r}} &= |\det F^{n-r}|^{\otimes-\alpha}. \end{aligned}$$

The distributional kernel of S_α is a $GL_n(\mathbb{R})$ -invariant generalized section \mathbb{S}_α over $Gr_r(\mathbb{R}^n) \times Gr_{n-r}(\mathbb{R}^n)$ of the line bundle $(L_\alpha^* \otimes |\omega_{Gr_r(\mathbb{R}^n)}|) \boxtimes M_\alpha$.

7.3 Remark. (1) We have also used the standard notation \boxtimes of the exterior product of two bundles. Namely if X_1, X_2 are two manifold and L_i , $i = 1, 2$, is a vector bundle over X_i then by definition

$$L_1 \boxtimes L_2 := p_1^* L_1 \otimes p_2^* L_2,$$

where $p_i: X_1 \times X_2 \rightarrow X_i$ is the natural projection, $i = 1, 2$.

(2) Recall that \mathbb{S}_α satisfies for any smooth section $f \in C^\infty(Gr_r(\mathbb{R}^n), L_\alpha)$

$$S_\alpha(f) = \int_{Gr_r(\mathbb{R}^n) \times Gr_{n-r}(\mathbb{R}^n)} \mathbb{S}_\alpha \cdot f,$$

and \mathbb{S}_α is uniquely characterized by this property.

Let P denote the stabilizer of $E_0^r = \text{span}\{e_1, \dots, e_r\}$. We will use the following theorem

7.4 Theorem (Frobenius descent, see e.g. [1, Theorem 4.2.3]). *Let a Lie group K act on a smooth manifold M . Let N be a smooth manifold with a transitive action of K . Let $\phi: M \rightarrow N$ be a K -equivariant map. Let $z \in N$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K . Let \mathcal{E} be a K -equivariant vector bundle over M .*

Then there exists a canonical isomorphism between K -invariant generalized sections of \mathcal{E} on M and K_z -invariant generalized sections of \mathcal{E} on M_z . Moreover, for any closed K -invariant subset $Y \subset M$, the generalized sections supported in Y are mapped to generalized sections supported in $Y \cap M_z$.

This isomorphism is the restriction to M_z . For the K -invariant generalized sections this restriction is well defined.

7.5 Corollary. *There is a natural one-to-one correspondence between $GL(V)$ -invariant generalized sections of the line bundle $(L_\alpha^* \otimes |\omega_{Gr_r(\mathbb{R}^n)}|) \boxtimes M_\alpha$ over $Gr_r(\mathbb{R}^n) \times Gr_{n-r}(\mathbb{R}^n)$ and P -invariant generalized sections of $M_\alpha \otimes (L_\alpha^* \otimes |\omega_{Gr_r(\mathbb{R}^n)}|)|_{E_0^r}$ over $Gr_{n-r}(V)$. Under this correspondence the support of the latter section is equal to the intersection of the support of the former one with the set $\{E_0^r\} \times Gr_{n-r}(V)$.*

7.6 Corollary. (1) *There is a well defined restriction map from the space of P -invariant generalized sections of the line bundle $M_\alpha \otimes (L_\alpha^* \otimes |\omega_{Gr_r(\mathbb{R}^n)}|)|_{E_0^r}$ over $Gr_{n-r}(\mathbb{R}^n)$ to the space of generalized functions on $\mathcal{N} \simeq Mat_{r \times r}(\mathbb{R})$. Generalized functions in the image of this map satisfy*

$$S(AXB) = |\det A \cdot \det B|^\alpha S(X) \quad (7.3)$$

for any $X \in Mat_{r \times r}(\mathbb{R})$, $A, B \in GL_r(\mathbb{R})$.

(2) *This map is injective.*

(3) *The support of the restriction of a section is equal to the intersection of the support of that section with \mathcal{N} .*

Proof. First restrict the generalized section to \mathcal{U} . This restriction is injective since \mathcal{U} intersects all P -orbits in $Gr_{n-r}(\mathbb{R}^n)$. Now consider the map $\phi : \mathcal{U} \rightarrow Mat_{r \times (n-2r)}(\mathbb{R})$ obtained by composing the diffeomorphism $\mathcal{U} \xrightarrow{\sim} Mat_{r \times (n-r)}(\mathbb{R})$ with the operation of taking the right $n - 2r$ columns. Let us define the subgroup $R \subset P$ consisting of matrices of the form

$$g = \begin{bmatrix} A_{r \times r} & 0 & 0 \\ 0 & I_{n-2r} & 0 \\ 0 & E_{r \times n-2r} & D_{r \times r} \end{bmatrix}. \quad (7.4)$$

Note that R preserves \mathcal{U} . Let R act on $Mat_{r \times (n-2r)}(\mathbb{R})$ by $g(Z) = DZ + E$, where $g \in R$ is given by (7.4). Note that this action is transitive, and the map $\phi : \mathcal{U} \rightarrow Mat_{r \times (n-2r)}(\mathbb{R})$ is R -equivariant. The space \mathcal{N} is the fiber of the zero matrix under ϕ , and the stabilizer in R of $0 \in Mat_{r \times (n-2r)}(\mathbb{R})$ is Q_0 . The corollary follows now from the Frobenius descent. Q.E.D.

Now we are going to study generalized functions on $Mat_{r \times r}(\mathbb{R})$ satisfying (7.3).

7.7 Proposition. *For any $\alpha \in \mathbb{C}$ the space of generalized functions S satisfying (7.3) is exactly one dimensional.*

The positivity of the dimension follows from existence of meromorphic continuation of the generalized function $|\det(\cdot)|^\alpha$ (this is a special case of a general result due to J. Bernstein [7] which says that for any polynomial P on \mathbb{R}^N the generalized function $|P|^\alpha$ has a meromorphic continuation to the whole complex plane). To prove uniqueness we will need some lemmas. Define

$$\mathcal{N}_k := \left\{ \begin{bmatrix} I_k & 0 \\ 0 & Y_{r-k, r-k} \end{bmatrix} \right\}.$$

Note that using the Frobenius descent any generalized function S satisfying (7.3) on the open subset consisting of matrices of rank at least k can be restricted to \mathcal{N}_k . This restriction \tilde{S} satisfies readily the same equation (7.3) but all matrices A, B, X have size $(r - k) \times (r - k)$.

7.8 Lemma. *If S satisfies (7.3) with parameter α then its Fourier transform satisfies (7.3) with parameter $-r - \alpha$.*

The proof is straightforward.

7.9 Lemma. *If S satisfies (7.3) with parameter α and is supported at 0 then S is proportional to*

$$(\det(\partial_{ij}))^{2k} \delta(X),$$

where ∂_{ij} is the partial derivative with respect to x_{ij} , $k = 0, 1, 2, \dots$, and $\alpha = -r - 2k \in -r - 2\mathbb{Z}_{\geq 0}$.

Proof. The Fourier transform $\mathbb{F}(S)$ is a polynomial. By Lemma 7.8 it satisfies (7.3) with parameter $-r - \alpha$. Hence $\mathbb{F}(S)$ must be a polynomial of even degree. This implies the lemma. Q.E.D.

7.10 Proposition. *Let $S \neq 0$ satisfy (7.3) with parameter α . Let $0 < l \leq r$ be an integer. Assume that $\text{supp}(S) = \{co - \text{rank} \geq l\}$. Then*

$$\alpha = -l, -l - 2, -l - 4, \dots$$

Proof. Let us restrict S to \mathcal{N}_{r-l} . This restriction \tilde{S} satisfies (7.3) with the same α but on the space of matrices of size l . Hence our lemma follows from Lemma 7.9 applied to \tilde{S} . Q.E.D.

We immediately deduce the following corollary.

7.11 Corollary. *If $S \neq 0$ satisfies (7.3) with $\alpha \neq -1, -2, -3, \dots$, then the support of S is equal to $\text{Mat}_{r \times r}(\mathbb{R})$. Moreover for such values of α the generalized function S is unique up to a multiplicative constant.*

Only uniqueness requires an explanation. On the open orbit the generalized function is clearly unique up to a multiplicative constant. Thus any two generalized functions S might be assumed to be equal on the open orbit. Then their difference is supported on smaller orbits, and hence vanishes. This proves Proposition 7.7 in this case. Hence it remains to prove the proposition only for $\alpha = -1, -2, -3, \dots$. By applying the Fourier transform and Lemma 7.8 we see that it remains to prove the proposition only for $\alpha = -1, -2, \dots, -[\frac{r}{2}]$.

7.12 Lemma. *The space of generalized functions S satisfying (7.3) with $\alpha = -1$ is one dimensional, and their support is equal to matrices of co-rank ≥ 1 (exactly).*

Proof. Let us restrict any such generalized function to \mathcal{N}_{r-1} . This restriction is an even (-1) -homogeneous generalized function on \mathbb{R} . Hence

it is a multiple of the delta function $\delta(x)$. Let us show that the kernel of this restriction map is zero. But the kernel consists of generalized functions supported on matrices of co-rank 2 and higher. By Proposition 7.10 the parameter $\alpha = -2, -3, -4, \dots$, i.e. not -1 ; this is a contradiction. Q.E.D.

The above discussion shows that we proved Proposition 7.7 for $r = 2, 3$, namely we have:

7.13 Corollary. *Let $r = 2, 3$. Then for any α the space of generalized functions satisfying (7.3) is one dimensional.*

Proof of Proposition 7.7. We prove by induction by r . For $r = 2, 3$ it is Corollary 7.13. Thus let us assume $r \geq 4$. It remains to consider the case $\alpha = -2, \dots, -[\frac{r}{2}]$.

Let us restrict S to \mathcal{N}_1 . This restriction is a generalized function on $Mat_{(r-1) \times (r-1)}(\mathbb{R})$ which satisfies (7.3) with the same α . By the induction assumption the space of such generalized functions is one dimensional. Hence it remains to show that the kernel of the restriction is zero. Indeed the kernel consists of generalized functions supported at 0. Then by Lemma 7.9, $\alpha \leq -r$. But $-r < -[\frac{r}{2}]$. Q.E.D.

7.14 Lemma. *If $S \neq 0$ satisfies (7.3) with $\alpha = -r, -r-2, -r-4, \dots$ then*

$$supp(S) = \{0\}.$$

Proof. By the uniqueness part we see that S is proportional to $(\det(\partial_{ij}))^{2k} \delta(X)$, $k = 0, 1, 2, \dots$. The result follows. Q.E.D.

7.15 Lemma. *If $S \neq 0$ satisfies (7.3) with $\alpha = -1, -2, \dots, -r$. Then*

$$supp(S) = \{co - rank \geq |\alpha|\}.$$

Proof. Let us restrict S to $\mathcal{N}_{r+\alpha} \simeq M_{|\alpha| \times |\alpha|}(\mathbb{R})$. S must be proportional to $\delta(X)$. If the coefficient of proportionality is not 0 then the lemma is proved. Let us assume that it vanishes. Then let us choose l such that $supp(S) = \{co - rank \geq l\}$. Thus $l > |\alpha|$. Then the restriction of S to $\mathcal{N}_{r-l} \simeq M_{l \times l}(\mathbb{R})$ does not vanish and is supported at 0. Hence $s = -l, -l-2, -l-4, \dots$, which contradicts to the inequality $l > |\alpha|$. Q.E.D.

7.16 Lemma. *If $S \neq 0$ satisfies (7.3) with $\alpha = -r-1, -r-3, -r-5, \dots$ then*

$$supp(S) = \{rk \leq 1\}.$$

Proof. Let us restrict S to $\mathcal{N}_1 \simeq M_{(r-1) \times (r-1)}(\mathbb{R})$. By Lemma 7.14 $\text{supp}(S|_{\mathcal{N}_1}) = \{0\}$. Equivalently $\text{supp}(S) \subset \{rk \leq 1\}$. It remain to show that $\text{supp}(S) \neq \{0\}$. Indeed otherwise we can apply Lemma 7.9 and get a contradiction. Q.E.D.

Let us summarize Proposition 7.7, Corollary 7.11, Lemma 7.14, Lemma 7.15, Lemma 7.16. We get

7.17 Theorem. (1) For any $\alpha \in \mathbb{C}$ the space of generalized functions S satisfying (7.3) is exactly one dimensional.

(2) The support of S is described completely by one of the following cases:

(a) If $\alpha \neq -1, -2, -3, \dots$ then $\text{supp}(S) = \text{Mat}_{r \times r}(\mathbb{R})$, i.e. is maximal.

(b) If $\alpha = -1, -2, \dots, -r + 1$, then $\text{supp}(S) = \{co - rank \geq |\alpha|\}$.

(c) If $\alpha \in -r - 2\mathbb{Z}_{\geq 0}$, then $\text{supp}(S) = \{0\}$.

(d) If $\alpha \in -r - 1 - 2\mathbb{Z}_{\geq 0}$, then $\text{supp}(S) = \{rk \leq 1\}$.

Now Theorem 1.8 of the introduction follows immediately from Lemma 7.5, Lemma 7.6, and Theorem 7.17 in the case $r = i$, namely $i \leq n - i$. It remains to consider the case $i > n - i$. This case easily follows from the previous one if one replaces $E \in Gr_i(V)$ and $F \in Gr_{n-i}(V)$ by their annihilators $E^\perp \in Gr_{n-i}(V^*)$ and $F^\perp \in Gr_i(V^*)$. Under this identification the α -cosine transform from $Gr_i(V)$ to $Gr_{n-i}(V)$ becomes the α -cosine transform from $Gr_{n-i}(V^*)$ to $Gr_i(V^*)$. Hence Theorem 1.8 is proved completely.

A Appendix. Proof of Theorem 6.12.

We start by recalling a few general basic facts on invariant differential operators on homogeneous spaces. Our main reference is [17, Ch. II, §4.2], where the case of connected groups is considered. Let G be a Lie group, let \mathfrak{g}_0 be the Lie algebra of G and let $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Let $H \subset G$ be a closed subgroup with Lie algebra \mathfrak{h}_0 and complexification \mathfrak{h} .

Let $D(G)$ denote the algebra of left G -invariant differential operators on G which are invariant with respect to all left translations. $D(G)$ is naturally isomorphic to the universal enveloping algebra $U(\mathfrak{g})$. Indeed the Lie algebra \mathfrak{g} acts on functions on G by right infinitesimal translations; they commute with the left ones. Hence we get a homomorphism of algebras $U(\mathfrak{g}) \rightarrow D(G)$. It is easy to see that it is an isomorphism.

Let us denote by $D_H(G)$ the algebra of differential operators on G which are left invariant under G and right invariant under H . Thus $D_H(G) \subset D(G)$. Under the above isomorphism $D(G) \simeq U(\mathfrak{g})$, $D_H(G)$ is isomorphic to the subalgebra $U(\mathfrak{g})^H$ of H -invariant elements.

Let $\pi: G \rightarrow G/H$ be the canonical map. The pull-back map is

$$\pi^*: C^\infty(G/H) \rightarrow C^\infty(G).$$

This gives a homomorphism of algebras

$$\tilde{\pi}: D_H(G) \rightarrow D(G/H)$$

which is uniquely characterized by the property $\pi^*(\tilde{\pi}(D)(f)) = D(\pi^*f)$ (here we have used that $D(\pi^*f)$ is invariant under right translations by H , hence is a pull-back under π of a function from $C^\infty(G/H)$).

A.1 Proposition. *Assume that there exists an $\text{Ad}(H)$ -invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (for example this holds provided H is compact). Then the homomorphism $\tilde{\pi}: D_H(G) \rightarrow D(G/H)$ is onto. Moreover its kernel is equal to $U(\mathfrak{g})^H \cap (U(\mathfrak{g}) \cdot \mathfrak{h})$ under the above described isomorphism $D_H(G) \simeq U(\mathfrak{g})^H$.*

Proof. For a connected group G the proposition was proved in [17], Ch. II, Theorem 4.6. We need to remove this assumption. In fact we will not reduce the result to the connected case, but give an independent proof.

First the algebra $\mathcal{D}(X)$ of differential operators with smooth coefficients on a manifold X has a canonical filtration by the differential operator's order $\{\mathcal{D}(X)_{\leq k}\}$; it is compatible with the product. We will prove first a more precise statement that $\tilde{\pi}$ is onto on subspaces of operator of order at most k for each k :

$$\tilde{\pi}: D_H(G)_{\leq k} \rightarrow D(G/H)_{\leq k}.$$

This is obviously true for $k = 0$. Using induction it suffices to show that $\tilde{\pi}$ is onto on all associated graded spaces

$$\tilde{\pi}: D_H(G)_{\leq k} / D_H(G)_{\leq k-1} \rightarrow D(G/H)_{\leq k} / D(G/H)_{\leq k-1}. \quad (\text{A.1})$$

We have

$$D_H(G)_{\leq k} / D_H(G)_{\leq k-1} \simeq \frac{U(\mathfrak{g})_{\leq k}^H}{U(\mathfrak{g})_{\leq k-1}^H} = \left(\frac{U(\mathfrak{g})_{\leq k}}{U(\mathfrak{g})_{\leq k-1}} \right)^H \simeq (\text{Sym}^k \mathfrak{g})^H. \quad (\text{A.2})$$

(Note that we have not used our assumption on H in the above isomorphisms.) On the other hand for any manifold X

$$\mathcal{D}(X)_{\leq k}/\mathcal{D}(X)_{\leq k-1} \simeq C^\infty(X, \text{Sym}^k(TX)),$$

where in the right hand side one has the space of smooth sections of the k th symmetric power of the tangent bundle TX . Hence for $X = G/H$ we get an imbedding

$$D(G/H)_{\leq k}/D(G/H)_{\leq k-1} \hookrightarrow (C^\infty(G/H, \text{Sym}^k(T(G/H))))^G = (\text{Sym}^k(\mathfrak{g}/\mathfrak{h}))^H \quad (\text{A.3})$$

Thus using (A.2) and (A.3) we obtain from (A.1) the maps

$$(\text{Sym}^k \mathfrak{g})^H \rightarrow D(G/H)_{\leq k}/D(G/H)_{\leq k-1} \hookrightarrow (\text{Sym}^k(\mathfrak{g}/\mathfrak{h}))^H. \quad (\text{A.4})$$

Thus it suffices to prove that the composition of these two maps is onto in order to show that the homomorphism $\tilde{\pi}$ is onto and thus finish the proof of the first part of the proposition. But the composed map is the canonical map

$$(\text{Sym}^k \mathfrak{g})^H \rightarrow (\text{Sym}^k(\mathfrak{g}/\mathfrak{h}))^H.$$

Its surjectivity follows immediately from the assumption on existence of $Ad(H)$ -invariant complement \mathfrak{m} of \mathfrak{h} in \mathfrak{g} .

It remains to describe the kernel of $\tilde{\pi}: D_H(G) \rightarrow D(G/H)$. By the Poincaré-Birkhoff-Witt theorem we have the isomorphism of vector spaces

$$U(\mathfrak{g}) \simeq \text{Sym}^\bullet(\mathfrak{m}) \oplus U(\mathfrak{g}) \cdot \mathfrak{h}. \quad (\text{A.5})$$

It is easy to see that an element $D \in U(\mathfrak{g})$ considered as a left G -invariant operator on functions on G vanishes on all right H -invariant functions if and only if under the isomorphism (A.5) it corresponds to an element of the form $U(\mathfrak{g}) \cdot \mathfrak{h}$. Q.E.D.

Now let us introduce some notation. For $0 \leq p \leq q$ let us denote

$$\begin{aligned} G_0 &= SO_0(p, q), \quad K_0 = SO(p) \times SO(q), \\ G_1 &= SO(p, q), \quad K_1 = S(O(p) \times O(q)), \\ G_2 &= O(p, q), \quad K_2 = O(p) \times O(q). \end{aligned}$$

Here $SO_0(p, q)$ denotes the connected component of the group $SO(p, q)$.

A.2 Proposition. *For any $i = 0, 1, 2$ the imbedding $K_i \hookrightarrow G_i$ induces an isomorphism on the groups of connected components.*

Proof. For $q = 0$ the statement is trivial, and for $q = 1$ it is well known. Let us assume that $q \geq 2$ and proceed by induction on q . For $i = 0, 1, 2$ let us denote by G'_i the group G_i with $(p, q - 1)$ instead of (p, q) , and K'_i the corresponding subgroup of G'_i . It is easy to see that the standard imbedding $K'_i \hookrightarrow K_i$ induces an isomorphism on π_0 . We have the commutative diagram:

$$\begin{array}{ccc} \pi_0(K'_i) & \longrightarrow & \pi_0(G'_i) \\ \downarrow & & \downarrow \\ \pi_0(K_i) & \longrightarrow & \pi_0(G_i), \end{array} \tag{A.6}$$

where the first horizontal map is an isomorphism by the induction assumption. It suffices to show that the second vertical arrow is an isomorphism. Notice that for $i = 0, 1, 2$

$$H := G_i/G'_i = \{x_1^2 + \cdots + x_q^2 - y_1^2 - \cdots - y_p^2 = 1\}.$$

Let us consider the smooth map $f: H \rightarrow \mathbb{R}^p \times S^{q-1}$ given by

$$f(x_1, \dots, x_q, y_1, \dots, y_p) = \left((y_1, \dots, y_p), \frac{(x_1, \dots, x_q)}{(x_1^2 + \cdots + x_q^2)^{\frac{1}{2}}} \right).$$

Clearly f is diffeomorphism. Hence for $q \geq 2$ the manifold H is connected. From the exact sequence

$$\pi_0(G'_i) \rightarrow \pi_0(G_i) \rightarrow \pi_0(H) = \{1\}$$

we get that $\pi_0(G'_i) \rightarrow \pi_0(G_i)$ is onto. From this and diagram (A.6) we deduce that the map

$$\pi_0(K_i) \rightarrow \pi_0(G_i)$$

is onto. But K_i and G_i have the same number of connected components, namely 1, 2, 4 for $i = 0, 1, 2$ respectively. Hence this map is an isomorphism. Q.E.D.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra of all three groups G_i , $i = 0, 1, 2$. Explicitly we choose them to be

$$\mathfrak{k} = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \mid X \in Mat_{p \times p}(\mathbb{C}), Y \in Mat_{q \times q}(\mathbb{C}) \text{ are anti-symmetric} \right\},$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0_{p \times p} & iW \\ -iW^t & 0_{q \times q} \end{bmatrix} \mid W \in Mat_{p \times q}(\mathbb{C}) \right\}.$$

Let us choose $\mathfrak{a} \subset \mathfrak{p}$ to be a maximal abelian subalgebra as follows:

$$\mathfrak{a} = \left\{ \begin{bmatrix} 0_{p \times p} & D & 0 \\ -D & 0_{p \times p} & 0 \\ 0 & 0 & 0_{q-p \times q-p} \end{bmatrix} \mid D \in Mat_{p \times p}(\mathbb{C}) \text{ is complex diagonal} \right\} \quad (\text{A.7})$$

Let us choose the basis $\lambda_1, \dots, \lambda_p$ in \mathfrak{a}^* as follows: the value of λ_j on the element of the above form with $D = i \cdot \text{diag}(a_1, \dots, a_p)$ is equal to a_j .

Then by [22], Ch. VII, §2.3, the (non-zero) roots Σ of \mathfrak{a} in \mathfrak{g} are

$$\pm (\lambda_i \pm \lambda_j), 1 \leq i < j \leq p, \text{ with multiplicity } 1; \quad (\text{A.8})$$

$$\pm \lambda_j, 1 \leq j \leq p, \text{ with multiplicity } q - p. \quad (\text{A.9})$$

Let us choose the positive roots as follows

$$\Sigma^+ = \{-(\lambda_i \pm \lambda_j) \mid 1 \leq i < j \leq p\} \cup \{-\lambda_j \mid 1 \leq j \leq p\}. \quad (\text{A.10})$$

The multiplicity of the root α will be denoted by m_α .

The half sum ρ of positive roots (counting multiplicities) is equal to

$$\rho = - \sum_{i=1}^p \left(\frac{n}{2} - i \right) \lambda_i. \quad (\text{A.11})$$

Finally let us define \mathfrak{n} to be the \mathbb{C} -span of positive root spaces; $\mathfrak{n} \subset \mathfrak{g}$ is a nilpotent subalgebra. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is the Iwasawa decomposition of the Lie algebra. Let $A, N \subset G_0$ be the real analytic subgroups obtained by the exponentiation of $\mathfrak{a} \cap so(p, q)$, $\mathfrak{n} \cap so(p, q)$ respectively.

A.3 Proposition. *For $i = 0, 1, 2$ we have the Iwasawa decomposition for groups G_i , namely the product map*

$$K_i \times A \times N \rightarrow G_i$$

is a diffeomorphism.

Proof. For $i = 0$ this is a classical result since the group G_0 is connected, see e.g. [17], Ch. VI, Theorem 5.1,. In general the result follows immediately from that case and Proposition A.2. Q.E.D.

Let us define

$$M_i := \{g \in K_i \mid \text{Ad}(g)(a) = a \text{ for any } a \in \mathfrak{a}\},$$

$$M'_i = \{g \in K_i \mid \text{Ad}(g)(\mathfrak{a}) \subset \mathfrak{a}\}.$$

Clearly $M_i \triangleleft M'_i$. The group $W_i := M'_i/M_i$ is finite, as will be seen below, and it is called the little Weyl group of the symmetric space G_i/K_i . Naturally $W_i \subset GL(\mathfrak{a})$. Let us describe M_i, M'_i explicitly; we leave the details of the elementary computations to the reader:

$$M_2 = \left\{ \left[\begin{array}{c|c|c} \varepsilon & 0 & 0 \\ \hline 0 & \varepsilon & 0 \\ \hline 0 & 0 & V \end{array} \right] \right\}, M'_2 = \left\{ \left[\begin{array}{c|c|c} \varepsilon S & 0 & 0 \\ \hline 0 & \delta S & 0 \\ \hline 0 & 0 & V \end{array} \right] \right\}$$

where S runs over all permutations of the standard basis in \mathbb{R}^p , ε, δ run over diagonal $p \times p$ matrices with ± 1 on the diagonal, $V \in O(q-p)$. Notice that

if $p = q$ then V disappears from both formulas. The matrix $\left[\begin{array}{c|c|c} \varepsilon S & 0 & 0 \\ \hline 0 & \delta S & 0 \\ \hline 0 & 0 & V \end{array} \right]$

acts on an element $i \cdot \text{diag}(x_1, \dots, x_p) \in \mathfrak{a}$ by

$$i \cdot \text{diag}(x_1, \dots, x_p) \mapsto (\varepsilon \delta^{-1}) \cdot i \cdot \text{diag}(x_{S(1)}, \dots, x_{S(p)}). \quad (\text{A.12})$$

From this the above description of M_2 is obvious.

It is clear that

$$M_i = M_2 \cap K_i, M'_i = M'_2 \cap K_i \text{ for } i = 0, 1.$$

Hence, in particular, $M'_i/M'_0 \subset K_i/K_0$. But using the above descriptions of M'_i it is easy to see that this inclusion is in fact an equality:

$$M'_i/M'_0 = K_i/K_0. \quad (\text{A.13})$$

The action of M'_2 on \mathfrak{a} induces a group homomorphism $M'_2 \rightarrow GL(\mathfrak{a})$ with kernel M_2 . Restricting it to M'_i , $i = 0, 1$ we get imbeddings

$$W_0 \subset W_1 \subset W_2.$$

Using these formulas it is easy to show that

$$W_0 = W_1 = W_2 = \{\pm 1\}^p \rtimes S_p \text{ for } p < q, \quad (\text{A.14})$$

$$W_0 = W_1 = \{\pm 1\}_{\text{even}}^p \rtimes S_p, W_2 = \{\pm 1\}^p \rtimes S_p \text{ for } p = q, \quad (\text{A.15})$$

where $\{\pm 1\}_{\text{even}}^p$ denotes the index 2 subgroup of $\{\pm 1\}^p$ consisting sequences of ± 1 with even number of minuses.

A.4 Proposition. *Let $K_i \cdot A \cdot N$ be the Iwasawa decomposition. Let dk, da, dn be Haar measures on K_i, A, N respectively (all of them are both left and right invariant). Then the Haar measure dg on G_i can be normalized so that*

$$dg = e^{2\rho(\log \bullet)} dk \cdot da \cdot dn,$$

where $\log: A \rightarrow \mathfrak{a} \cap \mathfrak{so}(p, q)$ is the inverse of the exponential map.

Proof. For G_0 this is proven in Proposition 5.1, Ch. I, §5 in [17]. The cases $i = 1, 2$ follow from this one by applying Proposition A.2 since G_0 is the connected component of the identity of G_i , and K_0 of K_i . Q.E.D.

A.5 Proposition (Harish-Chandra). *For $f \in C_c(G_i)$ we have*

$$\int_{G_i} f(g) dg = \int_{K_i N A} f(kna) dk \cdot dn \cdot da = \int_{AN K_i} f(ank) da \cdot dn \cdot dk.$$

Proof. For $i = 0$ this is Corollary 5.3, Ch. I, §5, in [17]. Combined with Proposition A.2 it implies other cases. Q.E.D.

A.6 Proposition. *For $H \in \mathfrak{a} \cap \mathfrak{so}(p, q)$, $a = \exp(H)$, let*

$$D(a) = \prod_{\alpha \in \Sigma^+} (\sinh(\frac{1}{2}\alpha(H)))^{m_\alpha}.$$

Then for a suitable normalization of invariant measure dg_A on G/A we have for any $a \in A$ such that $D(a) \neq 0$:

$$|D(a)| \cdot \int_{G/A} f(gag^{-1}) dg_A = e^{\rho(\log a)} \int_{K \times N} f(kank^{-1}) dk \cdot dn$$

for $f \in C_c(G_i)$ (the integrals on both sides are claimed to be absolutely convergent).

Proof. For G_0 this is proven in Proposition 5.6 in Ch. I, §5 of [17]. The cases $i = 1, 2$ follows from this one and Proposition A.2 by additional summation over connected components of G_i , or equivalently K_i . Q.E.D.

A.7 Proposition. *Let $f \in C_c(G_i)$ satisfy $f(kgk^{-1}) = f(g)$ for any $k \in K_i, g \in G_i$. Then the function*

$$F_f(a) := e^{\rho(\log a)} \int_N f(an) dn, \quad a \in A,$$

satisfies

$$F_f(a^s) = F_f(a), \quad \text{for each } a \in A, s \in W_i.$$

Proof. We just repeat the argument of Proposition 5.7 in Ch. I, §5 in [17]. By continuity it suffices to prove the identity for $D(a) \neq 0$. By Proposition A.6 we have

$$F_f(a) = |D(a)| \int_{G/A} f(gag^{-1}) dg_A.$$

We have to show that the right hand side of the last equality is W_i -invariant; we will do it in fact for any $f \in C_c(G_i)$.

W_i permutes the (restricted) roots Σ . Hence W_i preserves $|D(a)|$. Let $u \in M'_i$ be a representative of s . Since $uAu^{-1} = A$, the map $\phi: G_i/A \rightarrow G_i/A$ given by $\phi(xA) = uxu^{-1}A$ is a well defined diffeomorphism. The conjugation by u preserves Haar measures on G_i and on A since u is contained in the finite group W_i , hence ϕ preserves the G_i -invariant measure on G_i/A . We have, using $a^s = uau^{-1}$,

$$\begin{aligned} \int_{G_i/A} f(ga^s g^{-1}) dg_A &= \int_{G_i/A} f((ugu^{-1})a^s(ugu^{-1})^{-1}) dg_A \\ &= \int_{G_i/A} f(ugag^{-1}u^{-1}) \stackrel{\text{Prop. A.5}}{=} \int_{K_i \times N} f(uknan^{-1}k^{-1}u^{-1}) dk \cdot dn \\ &= \int_{K_i \times N} f(knan^{-1}k^{-1}) dk \cdot dn = \int_{G/A} f(gag^{-1}) dg_A. \end{aligned}$$

The proposition is proved. Q.E.D.

For $D \in U(\mathfrak{g}) \simeq D(G_i)$ let us denote by $D_{\mathfrak{a}} \in U(\mathfrak{a}) = D(A)$ the image of D under the projection

$$\tilde{\gamma}: U(\mathfrak{g}) = U(\mathfrak{n}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k}) \rightarrow U(\mathfrak{a})$$

which is just the quotient map under the subspace $\mathfrak{n} \cdot U(\mathfrak{g}) + U(\mathfrak{g}) \cdot \mathfrak{k}$. By Lemma 5.14 in Ch. II in [17] for any $\phi \in C_c(G_i)$ such that

$$\phi(n g k) = \phi(g) \text{ for any } g \in G_i, n \in N, k \in K_i$$

one has

$$(D\phi)|_A = D_{\mathfrak{a}}(\phi|_A). \quad (\text{A.16})$$

(Strictly speaking, in [17] the equality (A.16) is stated only for connected groups, i.e. for G_0 . But to prove (A.16) for G_i it suffices to restrict ϕ to G_0 .)

For $g \in G_i$ we denote by $A(g) \in \mathfrak{a} \cap \mathfrak{so}(p, q)$ the unique element such that $g \in N \cdot \exp(A(g)) \cdot K_i$.

A.8 Lemma. *For each linear functional $\nu: \mathfrak{a} \rightarrow \mathbb{C}$ the function*

$$\phi(g) := \int_{K_i} e^{\nu(A(kg))} dk$$

satisfies for any $D \in D_{K_i}(G_i) = U(\mathfrak{g})^{K_i}$

$$D\phi = D_{\mathfrak{a}}(\nu)\phi,$$

where $D_{\mathfrak{a}}(\nu) \in \mathbb{C}$ is defined as follows: we have

$$D_{\mathfrak{a}} \in D(\mathfrak{a}) = U(\mathfrak{a}) = S(\mathfrak{a}) = \mathbb{C}[\mathfrak{a}^*],$$

where $\mathbb{C}[\mathfrak{a}^]$ denotes complex polynomials on \mathfrak{a}^* ; thus $D_{\mathfrak{a}}$ can be evaluated at $\nu \in \mathfrak{a}^*$.*

Proof. Let $F(g) := e^{\nu(A(g))}$. Clearly $F(n g k) = F(g)$. Hence by (A.16) for any $a \in A$

$$(DF)(a) = (D_{\mathfrak{a}}F|_A)(a) = D_{\mathfrak{a}}(\nu) \cdot F(a). \quad (\text{A.17})$$

The functions DF and $D_{\mathfrak{a}}(\nu) \cdot F$ are both left N -invariant and right K_i -invariant, and by (A.17) coincide on A . Hence

$$DF = D_{\mathfrak{a}}(\nu) \cdot F.$$

Hence

$$\begin{aligned} (D\phi)(g) &= D_g \left(\int_{K_i} F(kg) dk \right) = \int_{K_i} (DF)(kg) dk = \\ &= D_{\mathfrak{a}}(\nu) \int_{K_i} F(kg) dk = D_{\mathfrak{a}}(\nu) \cdot \phi(g). \end{aligned}$$

Q.E.D.

A.9 Corollary. *The map $D_{K_i}(G_i) = U(\mathfrak{g})^{K_i} \rightarrow D(\mathfrak{a}) \simeq U(\mathfrak{a})$ given by*

$$D \mapsto D_{\mathfrak{a}}$$

is a homomorphism of algebras.

Proof. In the notation of the proof of Lemma A.8 for any $\nu \in \mathfrak{a}^*$ the corresponding function ϕ is an eigenfunction of any $D \in D_{K_i}(G_i)$ with the eigenvalue $D_{\mathfrak{a}}(\nu)$. Hence for any $D_1, D_2 \in D_{K_i}(G_i)$ this implies that

$$(D_1 \circ D_2)_{\mathfrak{a}}(\nu) = D_{1\mathfrak{a}}(\nu) \cdot D_{2\mathfrak{a}}(\nu).$$

Since this holds for any ν we get $(D_1 \circ D_2)_{\mathfrak{a}} = D_{1\mathfrak{a}} \circ D_{2\mathfrak{a}}$. The corollary follows. Q.E.D.

A.10 Theorem. *With $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_{\alpha} \cdot \alpha$ as above, for $\nu \in \mathfrak{a}^*$ let us denote*

$$\phi_{\nu}(g) := \int_{K_i} e^{(\nu+\rho)(A(kg))} dk, \quad g \in G_i.$$

Then $\phi_{s\nu} = \phi_{\nu}$ for each $s \in W_i$, where $(s\nu)(H) = \nu(s^{-1}H)$, $H \in \mathfrak{a}$.

Proof. Since $\phi_{\nu}, \phi_{s\nu}$ are bi-invariant under K_i , it suffices to prove for any $f \in C_c(G_i)$ which is bi-invariant under K_i that

$$\int_{G_i} \phi_{s\nu}(g) f(g) dg = \int_{G_i} \phi_{\nu}(g) f(g) dg. \quad (\text{A.18})$$

Under the decomposition $G_i = A \cdot N \cdot K_i$ by Proposition A.5 we have

$$dg = da \cdot dn \cdot dk.$$

Hence for $f \in C_c(G_i)$ being K_i -bi-invariant we have

$$\begin{aligned} \int_{G_i} \phi_{\nu}(g) f(g) dg &= \int_{K_i} dk \int_{G_i} e^{(\nu+\rho)(A(kg))} f(g) dg \\ &= \int_{G_i} e^{(\nu+\rho)(A(g))} f(g) dg = \int_A da \int_N e^{(\nu+\rho)(\log a)} f(an) dn \\ &= \int_A e^{\nu(\log a)} F_f(a) da, \end{aligned}$$

where $F_f(a) = e^{\rho(\log a)} \int_N f(an) dn$. But by Proposition A.7 $F_f(a^s) = F_f(a)$. Hence

$$\begin{aligned} \int_{G_i} \phi_{s\nu}(g) f(g) dg &= \int_A e^{(s\nu)(\log a)} F_f(a) da = \\ &= \int_A e^{\nu(\log a)} F_f(a^s) da = \int_{G_i} \phi_\nu(g) f(g) da. \end{aligned}$$

Q.E.D.

Let us define

$$\gamma: D_{K_i}(G_i) = U(\mathfrak{g})^{K_i} \rightarrow U(\mathfrak{a}) \quad (\text{A.19})$$

by $\gamma(D) = e^{-\rho} \circ D_{\mathfrak{a}} \circ e^\rho$, where e^ρ denotes the operator of shift in A by the element e^ρ . From Lemma A.8, Corollary A.9, Theorem A.10, and Proposition A.1 we immediately get

A.11 Corollary.

$$\gamma: D_{K_i}(G_i) \rightarrow U(\mathfrak{a})$$

is a homomorphism of algebras. Moreover its image is contained in $U(\mathfrak{a})^{W_i}$, and the kernel is equal to $U(\mathfrak{g})^{K_i} \cap (U(\mathfrak{g}) \cdot \mathfrak{k})$.

A.12 Remark. More explicitly, for any $D \in D_{K_i}(G_i)$, any $\nu \in \mathfrak{a}^*$, and any $s \in W_i$ one has

$$D_{\mathfrak{a}}(s\nu + \rho) = D_{\mathfrak{a}}(\nu + \rho),$$

and $\gamma(D)(\nu) = D_{\mathfrak{a}}(\nu + \rho)$.

Our first main goal is to prove

A.13 Theorem. For $i = 0, 1, 2$

$$\gamma: D_{K_i}(G_i) = U(\mathfrak{g})^{K_i} \rightarrow U(\mathfrak{a})^{W_i}$$

is an epimorphism of algebras with the kernel $U(\mathfrak{g})^{K_i} \cap (U(\mathfrak{g}) \cdot \mathfrak{k})$.

It remains only to prove that γ is onto. We will need a lemma. Let us denote for brevity

$$\mathfrak{a}_0 := \mathfrak{a} \cap so(p, q), \quad \mathfrak{p}_0 := \mathfrak{p} \cap so(p, q).$$

A.14 Lemma. *The restriction map*

$$C^\infty(\mathfrak{p}_0)^{K_i} \rightarrow C^\infty(\mathfrak{a}_0)^{W_i}$$

is an isomorphism of algebras.

Proof. For $i = 0$ this is Corollary 5.11(i) in Ch. II of [17]. For $i = 1, 2$ the injectivity follows from the case $i = 0$. Hence it remains to prove surjectivity.

Let $f \in C^\infty(\mathfrak{a}_0)^{W_i} \subset C^\infty(\mathfrak{a}_0)^{W_0}$. Then by case $i = 0$ there exists $\tilde{F} \in C^\infty(\mathfrak{p}_0)^{K_0}$ such that

$$\tilde{F}|_{\mathfrak{a}_0} = f.$$

Let us define a new function on \mathfrak{p}_0 (below the Haar measure dk on K_i is normalized to be probability measure):

$$F(x) = \int_{K_i} \tilde{F}(k^{-1}xk)dk = \frac{1}{|K_i/K_0|} \sum_{\sigma \in K_i/K_0} \tilde{F}(\sigma^{-1}x\sigma).$$

Clearly $F \in C^\infty(\mathfrak{p}_0)^{K_i}$. Let us show that $F|_{\mathfrak{a}_0} = f$.

By (A.13) one has $K_i/K_0 = M'_i/M'_0$. Then for any $a \in \mathfrak{a}_0$ we have

$$F(a) = \frac{1}{|M'_i/M'_0|} \sum_{\sigma \in M'_i/M'_0} \tilde{F}(\sigma^{-1}a\sigma) = f(a),$$

where the last equality is due to the facts that $\sigma^{-1}a\sigma \in \mathfrak{a}_0$ and f is W_i -invariant. Q.E.D.

A.15 Corollary. *The restriction of polynomials*

$$\mathbb{C}[\mathfrak{p}]^{K_i} \rightarrow \mathbb{C}[\mathfrak{a}]^{W_i}$$

is an isomorphism of algebras.

Proof. By Lemma A.14 only surjectivity has to be proven. Let $f \in \mathbb{C}[\mathfrak{a}]^{W_i}$ be a homogeneous polynomial. By Lemma A.14 there exists unique $F \in C^\infty[\mathfrak{p}_0]^{K_i}$ such that $F|_{\mathfrak{a}_0} = f$. F is also homogeneous of the same degree as f . But any infinitely smooth homogeneous function is a polynomial, i.e. $F \in \mathbb{C}[\mathfrak{p}]^{K_i}$. Q.E.D.

Proof of Theorem A.13. Let us define a linear map

$$\lambda: \text{Sym}(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

by $\lambda(X_1 \otimes \cdots \otimes X_m) := \frac{1}{m!} \sum_{\sigma \in S_m} X_{\sigma(1)} \cdots X_{\sigma(m)} \in U(\mathfrak{g})$. It is easy to see that λ is an isomorphism of vector spaces which commutes with the adjoint action of G_i .

Let \mathfrak{q} be the orthogonal complement of \mathfrak{a} in \mathfrak{p} with respect to the Killing form (recall that the restriction of the Killing form to \mathfrak{p}_0 is positive definite). Thus

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}.$$

Since the Killing form on \mathfrak{g} is G_i -invariant, \mathfrak{q} is M'_i -invariant. The Killing form induces identifications

$$\mathfrak{p}^* \simeq \mathfrak{p}, \quad \mathfrak{a}^* \simeq \mathfrak{a}$$

such that the first is K_i -equivariant, and the second is M'_i -equivariant. Under these identifications the restriction map

$$\mathbb{C}[\mathfrak{p}] \rightarrow \mathbb{C}[\mathfrak{a}]$$

becomes the projection

$$\tau: \text{Sym}(\mathfrak{p}) = \text{Sym}(\mathfrak{a}) \oplus \text{Sym}(\mathfrak{p}) \cdot \mathfrak{q} \rightarrow \text{Sym}(\mathfrak{a}).$$

By [17], Ch. II, formula (38), for any $p \in \text{Sym}(\mathfrak{p})^{K_0}$ one has

$$\text{degree}(\gamma(\lambda(p)) - \tau(p)) < \text{degree}(p). \quad (\text{A.20})$$

Let us fix now $t \in U(\mathfrak{a})^{W_i} = \text{Sym}(\mathfrak{a})^{W_i}$. We have to show that t belongs to the image of the homomorphism γ . We may assume that t is homogeneous. By Corollary A.15 and the above remarks on equivariant identifications, there exists unique $p \in \text{Sym}(\mathfrak{p})^{K_i}$ such that $\tau(p) = t$. Also p has the same degree as t . By (A.20)

$$\text{degree}(\gamma(\lambda(p)) - t) < \text{degree}(p). \quad (\text{A.21})$$

But $\lambda(p) \in U(\mathfrak{g})^{K_i}$. Continuing by induction in $\text{degree}(t)$ we prove the theorem. Q.E.D.

From Theorem A.13 and Proposition A.1 we immediately deduce

A.16 Corollary. *The epimorphism γ induces the isomorphism, also denoted by γ and called the Harish-Chandra isomorphism,*

$$\gamma: D(G_i/K_i) \xrightarrow{\sim} U(\mathfrak{a})^{W_i}.$$

Now let us start discussing the differential operators on compact groups. For $n = p + q$ let us denote

$$U_0 = U_1 = SO(n), U_2 = O(n).$$

Then $K_i \subset U_i$ for $i = 0, 1, 2$. Moreover U_0/K_0 is the Grassmannian of oriented p -planes in \mathbb{R}^n , while $U_1/K_1 = U_2/K_2$ is the Grassmannian of p -planes in \mathbb{R}^n . Notice that the complexified Lie algebra of U_i is naturally identified with $\mathfrak{g} = so(n, \mathbb{C})$. By Proposition A.1 the algebra $D(U_i/K_i)$ of U_i -invariant operators on U_i/K_i is naturally identified with quotient algebra $U(\mathfrak{g})^{K_i}/(U(\mathfrak{g})^{K_i} \cap (U(\mathfrak{g}) \cdot \mathfrak{k}))$. Hence we obtain an isomorphism

$$D(U_i/K_i) \simeq D(G_i/K_i). \quad (\text{A.22})$$

In order to finish the proof of Theorem 6.12 it remains to prove the following proposition.

A.17 Proposition. *Let $n = p + q$, $r = \min\{p, q\}$. Let $\mathcal{V} \subset L^2(O(n)/O(p) \times O(q))$ be an irreducible representation corresponding to a sequence m of even non-negative integers*

$$m = (m_1 \geq \dots \geq m_{r-1} \geq m_r \geq 0).$$

Let D be an invariant operator on $O(n)/O(p) \times O(q)$. Then D acts on \mathcal{V} by multiplication by a scalar $\gamma(D)(m + \bar{\rho})$, where $\bar{\rho}$ is the sequence given by

$$\bar{\rho}_i = n/2 - i, i = 1, \dots, r. \quad (\text{A.23})$$

Before the proof let us introduce some more notation. Let us agree as previously $0 < p \leq q$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ -invariant Cartan subalgebra containing \mathfrak{a} ; we do not need to specify it more explicitly. Thus

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b},$$

where $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{k}$. Moreover \mathfrak{h} can be chosen to be of the form

$$\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C},$$

where $\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0$ and $\mathfrak{a}_0 = \mathfrak{a} \cap so(p, q)$, $\mathfrak{b}_0 = i(\mathfrak{b} \cap (so(p) \times so(q)))$. Thus in any unitary representation of U_i the algebra \mathfrak{h}_0 acts by self-adjoint operators.

In particular the restriction to \mathfrak{h}_0 of any root of \mathfrak{g} with respect to \mathfrak{h} is a real valued linear functional on \mathfrak{h}_0 .

Let us choose the positive roots system of \mathfrak{g} with respect to \mathfrak{h} such that the restriction of these roots to \mathfrak{a} is equal to $\Sigma \setminus \Sigma^+ = -\Sigma^+$, where Σ is defined by (A.8)-(A.9), and Σ^+ is defined by (A.10). Let $\mathfrak{n}^+ \subset \mathfrak{g}$ be the \mathbb{C} -span of these roots. Thus $\mathfrak{h} \oplus \mathfrak{n}^+$ is a Borel subalgebra of \mathfrak{g} . Then we have

$$\theta(\mathfrak{n}) \subset \mathfrak{n}^+. \quad (\text{A.24})$$

Indeed for $\alpha \in \Sigma$ if

$$[a, x] = \alpha(a)x \text{ for any } a \in \mathfrak{a},$$

then applying the automorphism θ on both sides and using $\theta(a) = -a$ we get

$$[a, \theta(x)] = -\alpha(a)\theta(x).$$

This implies (A.24).

A.18 Lemma. *Let \mathcal{V} be an irreducible representation of the group $SO(n)$. Let $v \in \mathcal{V}$ be a highest weight vector, i.e. a non-zero vector satisfying*

$$(\mathfrak{h} \oplus \mathfrak{n}^+)(v) \subset \mathbb{C} \cdot v. \quad (\text{A.25})$$

Let $w \in \mathcal{V}$ be a non-zero $S(O(p) \times O(q))$ -invariant vector. Let (\cdot, \cdot) be a $SO(n)$ -invariant hermitian product. Then

- (1) $(v, w) \neq 0$.
- (2) $\mathfrak{b}(v) = 0$.

Proof.

Part (1) is a special case of Lemma 3 on p. 92 in [30].

To prove part (2) it suffices to show that for any $X \in \mathfrak{b}_0$ one has $X(v) = 0$. But by (A.25)

$$X(v) = c \cdot v$$

for some $c \in \mathbb{C}$. Since X is a self-adjoint operator

$$c(v, w) = (X(v), w) = (v, X(w)) = 0,$$

where the last equality is by assumption that w is $S(O(p) \times O(q))$ -invariant. Now part (1) implies part (2). Q.E.D.

Proof of Proposition A.17. Let us restrict our representation \mathcal{V} to $SO(n)$. If either $r < n/2$, or $r = n/2$ and $m_r = 0$ then \mathcal{V} is an irreducible $SO(n)$ -module. Otherwise \mathcal{V} is a direct sum of two irreducible $SO(n)$ -modules with highest weights $(m_1, \dots, m_{\frac{n}{2}-1}, m_{\frac{n}{2}})$ and $(m_1, \dots, m_{\frac{n}{2}-1}, -m_{\frac{n}{2}})$. Hence it suffices to prove that for any irreducible $SO(n)$ -module

$$\mathcal{V}_1 \subset L^2(SO(n)/S(O(p) \times O(q)))$$

with the highest weight

$$m_1 \geq \dots \geq m_{\frac{n}{2}-1} \geq m_{\frac{n}{2}} \geq 0$$

the action of $SO(n)$ -invariant operator D on \mathcal{V}_1 is multiplication by the scalar $\gamma(D)(m + \bar{\rho})$, where $\bar{\rho}$ is given by (A.23).

Let us choose $L \in U(\mathfrak{g})^{S(O(p) \times O(q))}$ to be a representative of D . Let (\cdot, \cdot) be an $SO(n)$ -invariant hermitian product which is assumed to be linear with respect to the second argument and anti-linear with respect to the first.

Let $v \in \mathcal{V}_1$ be a highest weight vector, i.e.

$$\mathfrak{n}^+ v = 0, \mathfrak{h} v \subset \mathbb{C} \cdot v. \quad (\text{A.26})$$

Let $\lambda \in \mathfrak{h}^*$ be the corresponding highest weight, i.e.

$$h(v) = \lambda(h)v \quad \forall h \in \mathfrak{h}.$$

Note that by Lemma A.18(2), λ vanishes on \mathfrak{b} , i.e.

$$\lambda \in \mathfrak{a}^* \subset \mathfrak{h}^*. \quad (\text{A.27})$$

Let $w \in \mathcal{V}_1$ be an $S(O(p) \times O(q))$ -invariant non-zero vector. By Lemma A.18(1)

$$(v, w) \neq 0. \quad (\text{A.28})$$

Hence it suffices to show that

$$(v, Dw) = (v, w) \cdot \gamma(D)(m + \bar{\rho}).$$

It is clear that for any $D_1 \in U(\mathfrak{g}) \cdot \mathfrak{k}$ one has $D_1 w = 0$ hence

$$(v, D_1(w)) = 0. \quad (\text{A.29})$$

Next we claim that for $D_2 \in \mathfrak{n} \cdot U(\mathfrak{g})$ one has

$$D_2^* v = 0. \quad (\text{A.30})$$

Hence it follows that

$$(v, D_2(w)) = 0. \quad (\text{A.31})$$

Indeed, to prove (A.30) let us choose $n \in \mathfrak{n}$. We may assume that $n \in \mathfrak{n} \cap \mathfrak{so}(p, q)$. Then using the Cartan decomposition for $\mathfrak{so}(p, q)$ consider

$$n = n_1 + n_2,$$

where $n_1 \in \mathfrak{k} \cap \mathfrak{so}(p, q)$, $n_2 \in \mathfrak{p} \cap \mathfrak{so}(p, q)$. The action on n_1 in \mathcal{V}_1 is anti-symmetric, while the action of n_2 on \mathcal{V}_1 is symmetric. Hence

$$n^* = -n_1 + n_2 = (-\theta)(n) \in \mathfrak{n}^+,$$

where the last inclusion is by (A.24). Hence (A.30) follows.

Then (A.29) and (A.31) imply that

$$\begin{aligned} (v, Dw) &= (v, D_{\mathfrak{a}} w) = ((D_{\mathfrak{a}})^* v, w) = \\ D_{\mathfrak{a}}(\lambda)(v, w) &= \gamma(D)(\lambda - \rho) \cdot (v, w), \end{aligned}$$

where the last equality is due to Remark A.12. Since $(v, w) \neq 0$ we get that D acts on \mathcal{V}_1 , and hence on \mathcal{V} , by the multiplication by the scalar $\gamma(D)(\lambda - \rho)$. We remind that here $\lambda \in \mathfrak{a}^* \subset \mathfrak{h}^*$, and $\rho \in \mathfrak{a}^*$ is the half sum of (restricted) roots from Σ^+ given by (A.10).

It remains to translate the last result into the combinatorial language. First we identify $\mathfrak{a}^* \xrightarrow{\sim} \mathbb{C}^p$ as follows. First identify \mathfrak{a} with \mathbb{C}^p as in (A.7) with the matrix D being diagonal with entries from \mathbb{C} . Dualizing this isomorphism we get an identification $\mathfrak{a}^* \simeq \mathbb{C}^p$.

If $\lambda \simeq \mathfrak{a}^* \subset \mathfrak{h}^*$ is the highest weight of this representation with the above choice of \mathfrak{n}^+ then it corresponds to a sequence

$$\begin{aligned} m_1 &\geq \cdots \geq m_{p-1} \geq |m_p| \text{ if } p = \frac{n}{2}, \\ m_1 &\geq \cdots \geq m_{p-1} \geq m_p \geq 0 \text{ if } p < \frac{n}{2}. \end{aligned}$$

Recall that in our case we have chosen $m_p \geq 0$ from the very beginning.

Furthermore under the above identification ρ corresponds to the sequence $\tilde{\rho}$ with

$$\tilde{\rho}_i = -(\frac{n}{2} - i), \quad i = 1, \dots, p.$$

Clearly $\tilde{\rho} = -\bar{\rho}$, where $\bar{\rho}$ is defined by (A.23). Theorem is proved. Q.E.D.

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