1 Lecture 8 (2/15/2011)

Key Terms: *Equivalence Principle, Contraposition, Cases, Contradiction.*

Remark 1 *We discuss some more principles of inference. To avoid confusion, we emphasize that these principles are not themselves "theorems"; rather they assert that the proof of some theorem(s) can be modified to yield the proof of other theorem(s). Thus we do not seek to "prove" the principles, but instead "establish" then by explaining the steps needed to carry out the desired modification.*

1.1 Equivalence Principle

The equivalence principle says: *If R and S are equivalent formulas then*

$$\{H_1, \ldots, H_n\} \vdash R \rightsquigarrow \{H_1, \ldots, H_n\} \vdash S$$

The main point is that if R and S are equivalent then $R \Rightarrow S$ is a tautology.

Now suppose $P_1, \ldots, P_{k-1}, R$ is a proof of Theorem 1 (on the left), then we can prove Theorem 2 (on the right) as follows:

$$P_1, \ldots, P_{k-1}, R, R \Rightarrow S, S$$

Why is this a valid proof of Theorem 2? Since Theorem 1 and Theorem 2 share the same hypotheses, all the statements $P_1, \ldots, P_{k-1}, R$ are still valid by the HTM rules. The statement $R \Rightarrow S$ is a tautology (as we observed above), and finally S is justified by modus ponens.

1.2 Contraposition Principle

Recall that the conditional $P \Rightarrow Q$ and its contrapositive $(\sim Q) \Rightarrow (\sim P)$ are equivalent formulas. This fact, combined with the deduction principle, yields the following *contraposition principle*:

$$\{H_1, \ldots, H_n, P\} \vdash Q \leftrightsquigarrow \{H_1, \ldots, H_n, \sim Q\} \vdash (\sim P) \tag{1}$$

We will establish one direction of this principle, namely

$$\{H_1, \ldots, H_n, \sim Q\} \vdash (\sim P) \rightsquigarrow \{H_1, \ldots, H_n, P\} \vdash Q$$

leaving the other direction as an exercise. We argue as follows:

$$\begin{aligned}
&\{H_1, \ldots, H_n, \sim Q\} \vdash (\sim P) \\
&\rightsquigarrow \{H_1, \ldots, H_n\} \vdash (\sim Q) \Rightarrow (\sim P) \ \text{(deduction)} \\
&\rightsquigarrow \{H_1, \ldots, H_n\} \vdash (P \Rightarrow Q) \ \ \text{(equivalence)} \\
&\rightsquigarrow \{H_1, H_2, \ldots, H_n, P\} \vdash Q \ \ \text{(deduction)}
\end{aligned}$$

1.3 Proof by Cases

The "proof by cases" principle is based on the fact that $(P \Rightarrow Q) \wedge ((\sim P) \Rightarrow Q)$ is equivalent to Q (check this). The precise statement of the principle is as follows:

$$\{H_1, \ldots, H_n, P\} \vdash Q, \{H_1, \ldots, H_n, \sim P\} \vdash Q \leftrightsquigarrow \{H_1, \ldots, H_n\} \vdash Q \qquad (2)$$

We will establish only the "$\rightsquigarrow$" direction. We argue as follows
Applying the deduction principle to the theorems on the left we get

$$\{H_1, \ldots, H_n\} \vdash (P \Rightarrow Q), \{H_1, \ldots, H_n\} \vdash ((\sim P) \Rightarrow Q)$$

By conjunctive inference we get

$$\{H_1, \ldots, H_n\} \vdash (P \Rightarrow Q) \wedge ((\sim P) \Rightarrow Q)$$

Now by equivalence we have

$$\{H_1, \ldots, H_n\} \vdash Q$$

Remark 2 *The two theorems on the left correspond to the "cases" where P is true and where P is false, respectively.*

1.4 Proof by Contradiction

Recall that a contradiction is a proposition that is always false; for example $Q \wedge (\sim Q)$. Also P and $(\sim P) \Rightarrow Q \wedge (\sim Q)$ are equivalent (check). This fact, combined with deduction gives the following important *contradiction principle*:

$$\{H_1, \ldots, H_n\} \vdash P \leftrightsquigarrow \{H_1, \ldots, H_n, (\sim P)\} \vdash Q \wedge (\sim Q) \qquad (3)$$

Once again we will establish only one direction, namely:

$$\{H_1, \ldots, H_n, (\sim P)\} \vdash Q \wedge (\sim Q) \rightsquigarrow \{H_1, \ldots, H_n\} \vdash P$$

We argue as follows:

$$\begin{aligned}
&\{H_1, \ldots, H_n, (\sim P)\} \vdash Q \wedge (\sim Q) \\
&\rightsquigarrow \{H_1, \ldots, H_n\} \vdash [(\sim P) \Rightarrow Q \wedge (\sim Q)] \ \text{(deduction)} \\
&\rightsquigarrow \{H_1, \ldots, H_n\} \vdash P \ \ \text{(equivalence)}
\end{aligned}$$

Remark 3 *In other words: to prove P from certain hypotheses, we assume $\sim P$ (along with the other hypotheses) and reach a contradiction.*

1.5 Exercises

1. Prove (using truth tables) that $(P \Rightarrow Q) \wedge ((\sim P) \Rightarrow Q)$ is equivalent to Q and also that P and $(\sim P) \Rightarrow Q \wedge (\sim Q)$ are equivalent

   The next three exercises are principles that need to be "established" as explained in Remark 1

2. $\{H_1, \ldots, H_n, P\} \vdash Q \rightsquigarrow \{H_1, \ldots, H_n, \sim Q\} \vdash (\sim P)$

3. $\{H_1, \ldots, H_n\} \vdash P \rightsquigarrow \{H_1, \ldots, H_n, (\sim P)\} \vdash Q \wedge (\sim Q)$

   [Hint: $Q \wedge (\sim Q)$ is equivalent to $P \wedge (\sim P)$.]

4. $\{H_1, \ldots, H_n\} \vdash Q \rightsquigarrow \{H_1, \ldots, H_n, P\} \vdash Q, \{H_1, \ldots, H_n, \sim P\} \vdash Q$

   [Hint: Does a proof remain valid if we add some hypotheses?]