1 Lecture 2 (1/20/2011)

Key Terms: *Conditional; Antecedent/Consequent; Inverse/Converse/Contrapositive; Biconditional*

Jason goes to his teacher to get back his test, and asks "How'd I do?" Reaching into the pile of tests, the teacher says,

If you scored 90 or more *then* you got an A.

The teacher's statement is a type of sentence called a *conditional* proposition, or sometimes just a conditional. It **is** a proposition, so we can decide whether it is true or false (by looking at Jason's test). It is false exactly when

Jason scored 90 or more *and* Jason did not get an A

otherwise the conditional is true. In particular, if Jason scored less than 90, then the conditional is true whether or not Jason got an A.

Definition 1 *If P and Q are propositions, then we can form the* conditional $P \Rightarrow Q$*, read varioulsy as "if P then Q", "Q if P", or "P implies Q". P is called the* antecedent *and Q is called the* consequent *of the conditional. The conditional is false precisely when P is false and Q is true.*

Theorem 2 *$(\sim P) \vee Q$ is equivalent to $P \Rightarrow Q$. $P \wedge (\sim Q)$ is a denial of $P \Rightarrow Q$.*

Proof

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$(\sim P) \vee Q$	$P \wedge (\sim Q)$
T	T	F	F	T	T	F
F	T	T	F	T	T	F
T	F	F	T	F	F	T
F	F	T	T	T	F	F

Definition 3 *Given $P \Rightarrow Q$ we define three other conditionals as follows:*

Inverse: $\sim P \Rightarrow \sim Q$; *Converse:* $Q \Rightarrow P$; *Contrapositive:* $\sim Q \Rightarrow \sim P$

Theorem 4 *The inverse and the converse of a proposition are equivalent. The proposition is equivalent to its contrapositive.*

Proof

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$\sim P \Rightarrow \sim Q$	$Q \Rightarrow P$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T	T	T	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F
F	F	T	T	T	T	T	T

Let us return to Jason and his teacher. Suppose the teacher says to Jason:

You got an A *if and only if* you scored 90 or more.

This statement is an example of a *biconditional* proposition. The biconditional is true precisely when the propositions "Jason got an A", "Jason scored 90 or more" have the same truth value, i.e. both are true or both are false.

Definition 5 *If P and Q are propositions, then the* biconditional *is $P \Leftrightarrow Q$, read variously as "P if and only if Q", "P iff Q". The biconditional is true precisely when P, Q are both true or they are both false.*

Theorem 6 *$P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.*

Proof

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
F	F	T	T	T	T

1.1 Exercises

1. Prove that

   (a) $\sim(P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$.
   (b) $\sim(P \wedge Q)$ is equivalent to $(\sim P) \vee (\sim Q)$.
   (c) $\sim(P \wedge Q)$ is equivalent to $P \Rightarrow (\sim Q)$.
   (d) $\sim(P \wedge Q)$ is equivalent to $Q \Rightarrow (\sim P)$.
   [The first two equivalences are called De Morgan's Laws.]

2. Prove that

   (a) $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$.
   (b) $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.
   (c) $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$.
   (d) $P \Rightarrow (Q \wedge R)$ is equivalent to $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.
   (e) $(P \vee Q) \Rightarrow R$ is equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

3. The *inverse* of a formula f is the formula $\bar{f}$ obtained by negating all variables. [For example if $f = P \wedge (Q \vee \sim R)$ then $\bar{f} = (\sim P) \wedge (\sim Q \vee R)$.] The *dual* $\tilde{f}$ of f is the negation of its inverse, thus $\tilde{f} = \sim\bar{f}$.

   Show using De Morgan's laws that the dual $\tilde{f}$ is equivalent to the formula obtained from f by replacing all $\wedge$ symbols by $\vee$ and vice versa.

4. Prove that if f is equivalent to g then

   a) $\bar{f}$ is equivalent to $\bar{g}$; b) $\sim f$ is equivalent to $\sim g$; c) $\tilde{f}$ is equivalent to $\tilde{g}$.

   [Hint: How are the truth tables of f and $\bar{f}$ *etc.* related to each other?]