1 Lecture 10 (2/22/2011)

Key Terms: *More proofs with quantifiers – $\exists x$ and $\exists! x$*

1.1 Proofs of $\exists x \; P(x)$

The quantified proposition $\exists x \; P(x)$ is the statement that

There is some element u in the universe U for which $P(u)$ is true.

How do we prove such a statement? The most direct way is to exhibit such an element. The proof usually starts with a statement of the form "Define $u = ...$", as is shown in the following example.

Theorem 1 *There exists a natural number that is both even and prime.*
[$\exists n \, (n$ is even$) \wedge (n$ is prime$)$; $U = \mathbb{N}$]

Proof.

1. Define $u = 2$.
2. Then u is a natural number.
3. Since $u = 2(1)$, u is even.
4. Since the only factors of u are $1, 2$, u is prime.
5. Therefore u is both even and prime.
6. Thus there exists a natural number that is both even and prime.

■

Here is a much more complicated example.

Theorem 2 *There exists a natural number whose fourth power is the sum of fourth powers of three other natural numbers.*
[$\exists a \exists b \exists c \exists d \; \left(a^4 + b^4 + c^4 = d^4\right)$; $U = \mathbb{N}$]

Proof. Define $a = 1,536,539$, $b = 2,682,440$, $c = 18,796.760$, $d = 20,615,673$. Then a, b, c, d are natural numbers. Moreover, $a^4 + b^4 + c^4 = d^4$ (This can be checked!). Therefore there exists a natural number whose fourth power is the sum of fourth powers of three other natural numbers. ■

Remark 3 *The main point of the above example is that it is not always "easy" to come up with the desired element(s) that establish the truth of $\exists x \; P(x)$. An alternative strategy that sometimes works is to argue by contradiction, namely to assume $\sim(\exists x P(x))$ and derive a false statement. In this context it is useful to remember that*

$$\sim(\exists x P(x)) \text{ is equivalent to } \forall x \, (\sim P(x)) \tag{1}$$

1.2 Proofs of $\exists! x \ P(x)$

The proposition $\exists! x \ P(x)$ is the statement that

There is exactly one element u in the universe U for which $P(u)$ is true.

The most direct way to prove this statement requires two steps

1. Existence: Find an element u in the universe such that $P(u)$ is true.
2. Uniqueness: Show that u is the *only* such element, by proving that if $P(v)$ is true then $v = u$.

Here is an example:

Theorem 4 *There exists a unique natural number that is both even and prime.*

Proof. (Existence) By Theorem 1 the natural number 2 is both even and prime.

(Uniqueness) Let x be a natural number. Suppose x is both even and prime. Since x is even, 2 divides x. Since x is prime, the only divisors of x are 1 and x. Therefore $2 = 1$ or $2 = x$. Since $2 \neq 1$, we must have $2 = x$. ■

1.3 Exercises

1. The pigeonhole principle says: "If m letters have been distributed into n pigeonholes, and $m > n$, then there exists a pigeonhole with 2 or more letters." Give a careful proof of this, following the strategy of Remark 3.

   [Hint: Here the universe is the set of pigeonholes; $P(x)$ is the property that pigeonhole x contains 2 or more letters; $\sim P(x)$ is the property that x contains 0 or 1 letters. By (1) the denial of $\exists x P(x)$ is $\forall x (\sim P(x))$. What relation between m and n does this imply? Where's the contradiction?]

2. If a, b, c, d are natural numbers with $\frac{a}{b} < \frac{c}{d}$ then there is a natural number n such that $\frac{a}{b} < \frac{n}{b+d} < \frac{c}{d}$

   [Hint: Of course n will depend in some manner on a, b, c, d and you have to "guess" which n will work, this is the creative part! For the more mundane part, to prove an inequality of the form $\frac{x}{y} < \frac{u}{v}$ with x, y, u, v positive, you simply need to show that $xv < yu$.]

3. In the notation of Exercise 2, suppose further that $bc - ad = 1$ and then prove that n is unique.

   [Hint: Show that the natural numbers bigger and smaller than the n you found in Exercise 2, will not work.]