Math 300, Lecture 25, Siddhartha Sahi

1 Onto and injective relations

Let A, B be sets and let $R \subseteq A \times B$ be a relation.

Definition 1 *We say that R is* onto *if $Rng\,(R) = B$.*

Example 2 *Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3, 4\}$ and consider the following relations*

$$R_1 = \{(a, 1), (b, 2)\}, R_2 = \{(a, 1), (a, 2), (b, 3), (c, 4)\}$$
$$R_3 = \{(a, 1), (b, 2), (c, 2)\}, R_4 = \{(b, 1), (b, 3), (c, 2)\}$$

Then $Rng(R_1) = Rng\,(R_3) = \{1, 2\}$, $Rng\,(R_4) = \{1, 2, 3\}$, therefore R_1, R_3, R_4 are not onto. However $Rng\,(R_2) = \{1, 2, 3, 4\} = B$, therefore R_2 is onto.

Definition 3 *We say that R is injective if*

$$\forall a, b, b' \left[(a, b) \in R \wedge (a, b') \in R \implies b = b'\right]$$

Example 4 *In the example above R_2 is not injective because $(a, 1) \in R_2, (a, 2) \in R_2$ but $1 \neq 2$. Similarly R_4 is not injective $(b, 1), (b, 3) \in R_4$, but $1 \neq 3$. However R_1, R_3 are both injective.*

Theorem 5 *If $R \subseteq A \times B$ and $S \subseteq B \times C$ are both injective, then so is $S \circ R$.*

Proof. Let A, B, C be sets and let $R \subseteq A \times B$, $S \subseteq B \times C$ be relations. Assume that R, S are both injective.
Since $R \subseteq A \times B$ and $S \subseteq B \times C$, we have $S \circ R \subseteq A \times C$.
Let $a \in A$ and $c, c' \in C$, and suppose $(a, c) \in S \circ R \wedge (a, c') \in S \circ R$.
Since $(a, c) \in S \circ R$, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
Since $(a, c') \in S \circ R$, there exists $b' \in B$ such that $(a, b') \in R$ and $(b', c') \in S$.
Since $(a, b) \in R \wedge (a, b') \in R$, and R is injective we get $b = b'$.
Thus we have $(b, c) \in S$ and, by substitution, $(b, c') \in S$.
Since S is injective, we get $c = c'$.
Therefore $S \circ R$ is injective. ■

Theorem 6 *If $R \subseteq A \times B$ and $S \subseteq B \times C$ are both* onto*, then so is $S \circ R$.*

Proof. Exercise. ■

2 More on functions

Recall that $R \subseteq A \times B$ is said to be a function if for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in R$. We can also reformulate this as follows:

Definition 7 *$R \subseteq A \times B$ is a function if*

1. *$Dom\,(R) = A$*
2. *$\forall a, a', b\,[(a, b) \in R \wedge (a', b) \in R \implies a = a']$*

Note that conditions 1 and 2 above are quite similar to the defnitions of "onto" and "injective". In fact we have the following result.

Theorem 8 *If $R \subseteq A \times B$, then R is a function iff R^{-1} is onto and injective.*

Proof. Exercise. ■

We can combine the previous three theorems to prove an important result.

Theorem 9 *If $R \subseteq A \times B$ and $S \subseteq B \times C$ are functions, then so is $S \circ R$.*

Proof. Suppose $R \subseteq A \times B$ and $S \subseteq B \times C$ are functions. Then by Theorem 8 R^{-1} and S^{-1} are onto and injective. By Theorems 5 and 6 $R^{-1} \circ S^{-1}$ is onto and injective. But $R^{-1} \circ S^{-1} = (S \circ R)^{-1}$. Therefore by Theorem 8 $S \circ R$ is a function. ■

2.1 Bijections

Definition 10 *A relation $R \subseteq A \times B$ is said to be a* bijection*, if R is a function that is both injective and onto.*

Theorem 11 *$R \subseteq A \times B$ is a bijection iff R^{-1} is a bijection.*

Proof. Exercise. ■

Theorem 12 *Let $\mathcal{S}$ be a collection of sets, (i.e. elements of $\mathcal{S}$ are themselves sets). Define a relation on $\mathcal{S}$ as follows*

$$\mathfrak{R} = \{(A, A') \in \mathcal{S} \times \mathcal{S} \mid \exists R \subseteq A \times A' \text{ such that } R \text{ is a bijection}\}$$

Then $\mathfrak{R}$ is an equivalence relation on $\mathcal{S}$.

Proof. Exercise. ■

2.2 Exercises

1. Prove Theorem 6.
2. Prove Theorem 8.
3. Prove Theorem 11.
4. Prove Theorem 12.