LECTURE 7 EXCERCISE SOLUTIONS

Problem. 1: Prove the equivalence of $R \wedge P \implies Q$ and $R \implies (P \implies Q)$.

Solution. What does it take for $R \wedge P \implies Q$ to be false? $R \wedge P$ must be true, and Q must be false. Thus, for $R \wedge P \implies Q$ to be false, both R and P must be true, and Q must be false. In any other case, the implication will be true.

What does it take for $R \implies (P \implies Q)$ to be false? R must be true, and $P \implies Q$ must be false. Thus, for $R \implies (P \implies Q)$ to be false, R must be true, P must be true, and Q must be false.

Thus, in both cases, the statement is false if P and R are true, and Q is false. In any other case, both statements are true. Thus they are equivalent.

There are also ways to prove this logically, but all but a handful people did it by truth table, and that's fine.

Common Problems. If people goofed, they usually goofed in performing some simple operating in their table. Again, it's important to show your work, otherwise you might be making stuff up out of thin air. It should also be noted that when you are proving the equivalence of propositions like this, you are not assuming anything about the values of the variables involved. That's the nice thing about the truth table approach - you are going through every possible value they might have. But what that means is that you need to be careful applying theorems to conclude things, because often theorems assume as hypotheses that certain things are true. And that may not be true in general.

Problem. 2: Prove the equivalence of $(P \implies R) \wedge (Q \implies R)$ and $P \vee Q \implies R$.

Solution. What does it take for $(P \implies R) \wedge (Q \implies R)$ to be false? Either $(P \implies R)$ is false, or $(Q \implies R)$ is false. Thus R must be false, and P or Q is true. In all other cases, the statement is true.

What does it take for $P \vee Q \implies R$ to be false? R must be false, and $P \vee Q$ must be true. In which case, P is true, or Q is true. In all other cases, the statement is true.

Again, we see that both statements have equivalent values for each possible assignment of the variables.

Common Problems. Again, there were a lot of the usual errors in truth-tabling. Most of the comments I made in the previous section apply here as well. One error I'm seeing here and there are people misinterpreting $P \vee Q \implies R$. The correct way to read that is as $(P \vee Q) \implies R$, instead of $P \vee (Q \implies R)$. The $\vee$ takes priority, and is evaluated first.

Problem. 3: Give a high-level argument for the reverse deduction principle, namely

$$\{H_1, ..., H_n\} \vdash (P \implies Q) \rightsquigarrow \{H_1, ..., H_n, P\} \vdash Q$$

Solution. By the strong citation principle, we now have $H_1 \wedge H_2 \wedge ... \wedge H_n \implies (P \implies Q)$. By the equivalence proven in Problem 1, we have $H_1 \wedge H_2 \wedge ... \wedge H_n \wedge P \implies Q$. Since $\{H_1, ..., H_n, P\} \vdash H_1 \wedge H_2 \wedge ... \wedge H_n \wedge P$, by the usual citation principle, we have

$$\{H_1, ..., H_n\} \vdash (P \implies Q) \rightsquigarrow \{H_1, ..., H_n, P\} \vdash Q$$

Alternatively, and this is interesting, starting with the hypotheses $\{H_1, ..., H_n\}$ and P, we can conclude first that $P \implies Q$, citing the first theorem, and then Q, since we have P, by modus ponens. Hence, $\{H_1, ..., H_n, P\} \implies Q$. This is a proof that justifies $\{H_1, ..., H_n, P\} \vdash Q$.

Common Problems. Lots to worry me, in this problem. To begin with, what is the problem asking you to show? It is saying, if assuming $\{H_1, ..., H_n\}$ can give a proof of $P \implies Q$, can you derive Q from assuming $\{H_1, ..., H_n, P\}$? One way to answer this is to prove that you can derive it (as in my first answer), and another is to actually give a proof of Q from $\{H_1, ..., H_n, P\}$, as in my second answer. One key thing here is that in either case, P needs to become a hypothesis, something you're assuming for the sake of the problem. A related problem I saw was people introducing hypotheses that weren't justified. All you have to work with is $\{H_1, ..., H_n, P\}$, and the fact that $\{H_1, ..., H_n\} \vdash (P \implies Q)$

Another thing I saw a lot of was people introducing statements they claimed were tautologies but weren't. A tautology is true for *all* truth values of its variables. Keep that in mind.

Lastly, there is the question of justification again. Make sure that your conclusions are justified. If you have an expression in one form, and then in the next line change it to another, make sure the reason is indicated. And make sure it is the right reason.

Post-lastly, please please please try to make it as easy as possible for me to follow your work across the page. If you have huge blocks of text that aren't contributing, cross them out, or indicate it somehow. Give me some kind of narrative thread to follow.

Problem. 4: Give a high-level argument for the following,

$$\{H_1, ..., H_n, P \vee Q\} \vdash R \rightsquigarrow \{H_1, ..., H_n, P\} \vdash R, \{H_1, ..., H_n, Q\} \vdash R$$

Solution. Jumping in with the rightwards deduction principle, we have

$$\{H_1, ..., H_n, P \vee Q\} \vdash R \rightsquigarrow \{H_1, ..., H_n\} \vdash (P \vee Q \implies R)$$

Using the equivalence in Problem 2, we know $P \vee Q \implies R$ is equivalent to $(P \implies R) \wedge (Q \implies R)$. Thus we have established

$$\{H_1, ..., H_n\} \vdash (P \implies R) \wedge (Q \implies R)$$

Using the tautology that $A \wedge B \implies B$, we get both of the following

$$\{H_1, ..., H_n\} \vdash (P \implies R)$$
$$\{H_1, ..., H_n\} \vdash (Q \implies R)$$

Using the leftwards deduction principle, we reach the following

$$\{H_1, ..., H_n, P\} \vdash R$$
$$\{H_1, ..., H_n, Q\} \vdash R$$

Thus, chaining all our $\rightsquigarrow$s together,

$$\{H_1, ..., H_n, P \vee Q\} \vdash R \rightsquigarrow \{H_1, ..., H_n, P\} \vdash R$$
$$\{H_1, ..., H_n, P \vee Q\} \vdash R \rightsquigarrow \{H_1, ..., H_n, Q\} \vdash R$$

And combining them in the usual fashion,

$$\{H_1, ..., H_n, P \vee Q\} \vdash R \rightsquigarrow \{H_1, ..., H_n, P\} \vdash R, \{H_1, ..., H_n, Q\} \vdash R$$

Common Problems.