LECTURE 24 EXCERCISE SOLUTIONS

Problem. 1: Prove $R \subset A \times B$ is a function iff $\forall a \in A, \exists! b \in B, (a, b) \in R$.

Solution. Suppose R is a function, and consider any $a \in A$. By the definition of function, $Dom(R) = A$, so $a \in Dom(R)$. Further, by the definition of a function, there exists a unique $b \in B$ such that $(a, b) \in R$. Hence, for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in R$.

Suppose that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in R$. Since for all $a \in A$, $(a, b) \in R$ for some $b \in B$, we have that $a \in Dom(R)$, so $A \subset Dom(R)$. By definition, $Dom(R) \subset A$. Hence we have that $Dom(R) = A$. Further, for all $a \in Dom(R)$, since $Dom(R) = A$, we have by assumption that there is a unique $b \in B$ such that $(a, b) \in R$.

Hence, both directions of the iff hold.

Common Problems. The hard part of this problem is likely knowing exactly what you need to prove for each direction. For the first direction, we have that for all $a \in Dom(R)$ there exists a unique b, etc, and you need to prove it for all $a \in A$. For the other direction, you need to prove two things, that $Dom(R) = A$, and that the for all/existence statement holds over the domain of R, not just A. It may feel like you're repeating yourself a lot, but thoroughness is rarely a bad thing.

Problem. 2: If S is a subset of $\mathbb{R} \times \mathbb{R}$, then S is a function iff the vertical line of the form $x = constant$ meets S in exactly one point.

Solution. Suppose that S is a function. Then the domain of S is $\mathbb{R}$. Also, for any real number x, there is a unique y such that $(x, y) \in S$. So consider the vertical line given by $x = K$ for some constant K. This is all points of the form (K, y) for all $y \in \mathbb{R}$. Since S is a function, $Dom(S) = \mathbb{R}$, so $K \in Dom(S)$, and there is a unique y' such that $(K, y') \in S$, and no other points of the form (K, y) are in S. Hence, the vertical line and S intersect at only one point, given by (K, y) for that unique y'.

Suppose that any vertical line $x = K$ for some constant K meets S at exactly one point. That is to say, for any $K \in \mathbb{R}$, there exists exactly one point (K, y) such that $(K, y) \in S$. First, since such a point exists for any $K \in \mathbb{R}$, we have that $\mathbb{R} \subset Dom(S)$. Since $Dom(S) \subset (R)$ by definition, we have that $Dom(S) = \mathbb{R}$. Further, consider any $K \in Dom(S)$. By assumption, there exists a unique point of the form $(K, y) \in S$. Hence, for any $K \in Dom(S)$, there exists a unique $y \in \mathbb{R}$ such that $(K, y) \in S$. Hence, S is a function.

Common Problems. Again, the hard part is likely knowing what you need to prove. In the first direction, you need to show that any vertical line does intersect S, and the point of intersection is unique. In the second direction, you need to show that the domain of S is $\mathbb{R}$, which follows since you can take 'any vertical line', and then you need to show that for every value in the domain of S, there is a unique value in the range, which follows since the vertical line intersection is unique.