LECTURE 23 EXCERCISE SOLUTIONS

Problem. 1: Prove that if $S_1, S_2, ..., S_n$ are subsets of A, then $S_1 \cup S_2 \cup ... \cup S_n \subset A$.

Solution. Let $x \in S_1 \cup S_2 \cup ... \cup S_n$. If x is in the union of all those sets, then there must be at least one S_i such that $x \in S_i$. If not, then x certainly couldn't be in their union. By assumption, $S_i \subset A$. Hence, $x \in A$. Therefore, $S_1 \cup S_2 \cup ... \cup S_n \subset A$.

Common Problems. When asked to prove one set is contained in another, it is a very common approach to simply take an arbitrary element in the one set, and follow your nose until you can show it is contained in the other.

Problem. 2: Suppose that $\pi = \{S_1, S_2, ..., S_n\}$ satisfies 1) $S_1 \cup ... \cup S_n = A$ and 2) if $i \neq j$, then $S_i \cap S_j = \emptyset$. Prove that π is a partition of A.

Solution. It suffices to show that for any $x \in A$, there exists a unique S_i such that $x \in S_i$. This is the definition of a partition. So consider let x be an element of A. Because $S_1 \cup ... \cup S_n = A$, $x \in S_1 \cup ... \cup S_n$. Hence x must be in at least one of the sets $S_1, ..., S_n$. This proves existence; we must further show uniqueness. Suppose that $x \in S_i$ and $x \in S_j$. Therefore, $x \in S_i \cap S_j$. By assumption, if $i \neq j$, the intersection is empty. Since it is not empty (it contains x), it must be that $i = j$, hence $S_i = S_j$.
We have therefore shown that for any element in A, there exists some S_i that contains it, and that S_i is unique. Therefore, π is a partition of A.

Common Problems. Existence is relatively straightforward. The key to this problem, as I see it, is uniqueness. But considering uniqueness almost leads directly to considering the intersection, at which point assumption two comes into play.

Problem. 3: Let $\pi = \{S_1, S_2, ..., S_n\}$ be a partition of A. Let R be a relation such that $(a, b) \in R$ if there exists some S_i such that $a \in S_i$ and $b \in S_i$. Prove that R is an equivalence relation, and that $A/R = \pi$, or that the equivalence classes of R are given by π.

Solution. First, showing that R is an equivalence relation. We must show that it is reflexive, symmetric, and transitive. Let $a \in A$. Since π is a partition, $a \in S_i$ for some S_i. Hence, there is some S_i such that $a \in S_i$ and $a \in S_i$. Hence, $(a, a) \in R$. Therefore, R is reflexive.

Suppose that $(a, b) \in R$. Then there exists some S_i such that $a \in S_i$ and $b \in S_i$. This is equivalent to $b \in S_i$ and $a \in S_i$. Hence, $(b, a) \in R$. Therefore, R is symmetric.

Suppose that $(a, b) \in R, (b, c) \in R$. Since $(a, b) \in R$, there exists some set S_j such that $a \in S_j$ and $b \in S_j$. Since $(b, c) \in R$, there exists some set S_i such that $b \in S_i$ and $c \in S_i$. Note that $b \in S_j$ and $b \in S_i$. Since π is a partition, b is in a unique set in π, therefore $S_i = S_j$. Hence, we have S_i such that $a \in S_i, b \in S_i, c \in S_i$. Hence, (I say hence a lot) $a \in S_i$ and $c \in S_i$. Therefore, $(a, c) \in R$. Therefore, R is transitive.

Then, the equivalence classes. Let E be an equivalence class of R. Consider any $x \in E$. Since $x \in A$, and π is a partition of A, $x \in S_i$ for some S_i. So consider any $y \in S_i$. Clearly, we have an S_i such that $x \in S_i$ and $y \in S_i$, therefore $(x, y) \in R$. Therefore, $y \in E$. Hence, $S_i \subset E$. Further, consider any $z \in E$. Since $x \in E$, $(x, z) \in R$. Therefore, there exists some S_j such that $x \in S_j$ and $z \in S_j$. However, since $x \in S_i$ and $x \in S_j$, by the uniqueness property of partitions, we know that $S_i = S_j$. Hence, $z \in S_i$. Therefore, $E \subset S_i$. Since $E \subset S_i, S_i \subset E$, we have that $E = S_i$.

This shows us that each equivalence class is one of the sets in π. Further, the above also shows that, taking any S_k and taking $x \in S_k$, the equivalence class associated with x must be S_k. Hence, every set in π is an equivalence class. Since every set in π is an equivalence class, and every equivalence class is a set in π, the two must be equal: $A/R = \pi$.

Common Problems. The hard part of this problem is, I think showing that the set of equivlance classes is π. There are probably many ways to do this, but it should always take something like this form: showing that $\pi \subset A/R$ (any S_i is an equivalence class), and then that $A/R \subset \pi$ (any equivalence class is some S_i), therefore $\pi = A/R$.