LECTURE 22 EXCERCISE SOLUTIONS

Problem. 1: Let $A = \{1, 2\}$. List the ordered pairs and draw the digraph for a relation on A, for each of the following properties.
a) Not reflexive, not symmetric, and not transitive.
b) Reflexive, not symmetric, and not transitive.
c) Not reflexive, symmetric, and not transitive.
d) Reflexive, symmetric, and not transitive.
e) Not reflexive, not symmetric, and transitive.
f) Reflexive, not symmetric, and transitive.
g) Not reflexive, symmetric, and transitive.
h) Reflexive, symmetric, and transitive.
8 Points, 1 each.

Solution. Note to begin with, that the empty relation is both symmetric and transitive. The conditions for symmetry and transitivity must hold for all elements in the relation, and if there are no elements in the relation, they certainly do hold. The empty set causes all kinds of trouble that way.

Similarly, a relation with one pair is necessarily transitive. Do you see why?

Note the following as well. With two elements in our set, there are four possible pairs to consider: $(1, 1), (1, 2), (2, 2), (2, 1)$.

If the relation is *not* symmetric, it must contain one of $(1, 2), (2, 1)$, but not both.

a) Since the relation we want is not symmetric, it cannot be empty. Since the relation we want is not transitive, it cannot have one pair. Therefore, it either has two pairs, three pairs, or four pairs. If it had four pairs, it would necessarily contain $(1, 1), (2, 2)$, making it reflexive. Therefore it cannot have four pairs. Therefore, it either has two pairs or three pairs. If it had three pairs, since it cannot contain $(1, 1), (2, 2)$ together, it is either $\{(1, 1), (1, 2), (2, 1)\}$ or $\{(1, 2), (2, 2), (2, 1)\}$. However, both of these are symmetric, so the relation we are after cannot contain three pairs. Therefore it must contain two pairs. Without loss of generality, consider the relation $\{(1, 1), (1, 2)\}$. This is in fact transitive, since for any two pairs of the form $(a, b), (b, c)$, the relation contains the pair (a, c). Indeed, any of the possible remaining two-pair relations will be transitive. Therefore, the relation we are after cannot contain only two pairs. Since we've ruled out every possible number of pairs the relation can have, the relation cannot exist.

b) Since reflexive, the relation contains $(1,1),(2,2)$. Therefore, the relation contains either two, three, or four pairs. Since not symmetric, it contains either $(1,2)$ or $(2,1)$, but not both. Therefore the relation contains three pairs. Without loss of generality, consider the relation $\{(1,1),(2,2),(1,2)\}$. Again, this relation is transitive since for any possible $(a,b),(b,c)$ pair of pairs in the relation, (a,c) is in the relation. Indeed, the same logic rules out any of the remaining possible three pair relations. Again, this gives a contradiction, and no such relation exists.

c) $\{(1,2),(2,1)\}$. Not reflexive for the obvious reason, symmetric, and not transitive since it doesn't contain $(1,1)$.

d) Reflexive means the relation contains $(1,1),(2,2)$. This is transitive, so it must contain something more. To maintain symmetry, it must contain $(1,2),(2,1)$, so the relation contains all four possible pairs. Unfortunately, the relation containing all four pairs is transitive. Therefore, no such relation can exist.

e) $\{(1,2)\}$. Not reflexive, for the obvious reasons, not symmetric since it doesn't contain $(1,2)$, and transitive since for every $(a,b),(b,c)$ pair of pairs (of which there are none), it contains (a,c).

f) $\{(1,1),(1,2),(2,2)\}$. Reflexive, not symmetric, and for every $(a,b),(b,c)$ pair, it contains (a,c), so transitive.

g) $\{(1,1)\}$. Not reflexive, since it doesn't contain $(2,2)$, definitely symmetric, and for every $(a,b),(b,c)$ pair of pairs (of which there are one), it contains (a,c), so it is transitive.

h) $\{(1,1),(2,2)\}$ would suffice, as would the set of all four pairs, $\{(1,1),(1,2),(2,2),(2,1)\}$.

Common Problems. I didn't expect an explanation or justification for each, I only graded on whether the answer given was correct.

Problem. 2: Let R be a relation on A, and prove the following.
a) R is reflexive iff $I_A \subset R$.
b) R is symmetric iff $R^{-1} \subset R$.
c) R is transitive iff $R \circ R \subset R$.
6 Points, 2 each.

Solution. a) R is reflexive $\iff \forall a \in A, (a,a) \in R \iff \forall (a,a) \in I_A, (a,a) \in R \iff I_A \subset R$.

b) Noting that $(b,a) \in R$ is equivalent to $(a,b) \in R^{-1}$, we can say the following: R is symmetric $\iff \forall (a,b) \in R, (b,a) \in R \iff \forall (b,a) \in R^{-1}, (b,a) \in R \iff R^{-1} \subset R$.

c) Note that R is transitive if $\forall a,b,c \in A, ((a,b) \in R \wedge (b,c) \in R) \implies (a,c) \in R$. Also, by the definition of composition, for any $(a,c) \in R \circ R$, there is some $b \in A$ such that $(a,b),(b,c) \in R$. Therefore, we can say the following:

Assume R is transitive. For any $(a,c) \in R \circ R$, we have some $b \in A$ such that $(a,b),(b,c) \in R$. By transitivity, $(a,c) \in R$. Therfore, $R \circ R \subset R$.

Assume $R \circ R \subset R$. Assume that $(a,b),(b,c) \in R$. Then $(a,c) \in R \circ R$. Since $R \circ R \subset R$, $(a,c) \in R$. Therefore, $\forall a,b,c \in A, ((a,b) \in R \wedge (b,c) \in R) \implies (a,c) \in R$. Therefore, R is transitive.

Common Problems. Since these were all $\iff$ statements, I'd intended each to be 2 points, one point for each direction. Since most people did all three as a chain of $\iff$ statements, effectively proving both directions at once, it really became a question of one point for the right idea, and one point for how well I thought you accomplished what you were trying to prove.

The problem, if I may take a brief aside, with doing a proof by a chain of $\iff$ statements, is that your proof often becomes just a chain of symbols, and it is very difficult to see why you are doing what you are doing, and whether or not it is justified. I strongly encourage some amount of words to at least outline what you are doing, and justify why you can do that. Words always help.

To that end, points were lost for the most part because I couldn't see the connections, how you were justifying one step to the next. Many people seemed to have the right sort of idea, but it got lost in a jumble of symbols when they tried to put it in the proof, and I had trouble following what you were trying to do. This was especially common in the third proof, when in the jump from transitivity to the condition on $R \circ R$, people were combining symbols and statements in a way I don't think applied to the problem, if they were true at all. Similarly in the second, the jump from talking about things in R to talking about things in R^{-1} was often not justified, or presented in such a way I couldn't really tell what you were trying to do. Part of the problem in that case, I feel, is that if R is symmetric, it is actually true that $R^{-1} = R$, and most people almost proved that, but weakened their proof in the last minute to make it $R^{-1} \subset R$, without making it clear what they were doing.

Problem. 3: Let m be a fixed natural number, and define $R_m = \{(x, y) | \exists k \in \mathbb{Z}, x - y = km\}$. Prove R_m is an equivalence relation, and that the set of equivalence classes of R_m is given by m distinct classes, $Z_m = \{\overline{0}, \overline{1}, ..., \overline{m-1}\}$.
6 Points total, 1 for each property of an equivalence relation, 3 for showing the m distinct classes.

Solution. Note that $x - x = 0 = 0m$ for any integer x. Therefore, $(x, x) \in R_m$ for all $x \in \mathbb{Z}$, so R_m is reflexive.

Assume that $(x, y) \in R_m$. Then there exists some integer k such that $x - y = km$. Note then, that $y - x = (-k)m$, so $y - x$ is an integer multiple of m as well. Therefore, $(y, x) \in R_m$, and R_m is symmetric.

Assume that $(x, y), (y, z) \in R_m$. Then there exist integers k, j, such that $x - y = km, y - z = jm$. In that case, $x - z = (x - y) + (y - z) = km + jm = (k + j)m$. Therefore, $x - z$ is an integer multiple of m as well, and $(x, z) \in R_m$. Therefore, R_m is transitive.

Being reflexive, symmetric, and transitive, R_m is an equivalence relation.

Consider any integer n. By a previous theorem, there exist unique integers q, r with $0 \leq r \leq m - 1$ such that $n = qm + r$. In that case, $n - r = qm$, and we see that $(n, r) \in R_m$, or that n is in the equivalence class $\overline{r}$, and $\overline{r} \in \{\overline{0}, \overline{1}, ..., \overline{m-1}\}$. Hence, any integer is in one of these m equivalence classes. And clearly each of those equivalence classes are nonempty, just looking at the integers $0, 1, 2, ..., m - 1$.

It remains to show that the m equivalence classes given above are distinct. Suppose that $\overline{r} = \overline{s}$, $0 \leq r < m, 0 \leq s < m$. We may assume without loss of generality that $r \geq s$. Since $\overline{r} = \overline{s}$, we see that $r \in \overline{s}$, therefore, $r - s$ is an integer multiple of m. However, $0 \leq r - s \leq r < m$. The only integer multiple of m less than m and greater than or equal to 0 is 0. Therefore, $r - s = 0$, or $r = s$. Therefore, each of the m equivalence classes are distinct.

Common Problems. Many people lost points for only doing half the problem, either not proving that it is an equivalence relation, or not proving that the classes were distinct.