LECTURE 21 EXCERCISE SOLUTIONS

Problem. 1: Give the digraphs of the following relations on the set $\{1, 2, 3\}$.

a) $=$
b) $S = \{(1, 3), (2, 1)\}$
c) $\leq$
d) S^{-1}, where $S = \{(1, 3), (2, 1)\}$
e) $\neq$
f) $S \circ S$, where $S = \{(1, 3), (2, 1)\}$
1 Point each, 6 total.

Solution. For the digraph of a relation R, if $(a, b) \in R$, then the digraph will have an arrow pointing *from* a and going *to* b. I don't know how to embed images in these things, so I will simply list the edges you should've had.

a) $=$: The relation can be listed out fully as $' =' = \{(1, 1), (2, 2), (3, 3)\}$. The only elements of the relation are of the form (a, a), because any number is only equal to itself. The digraph has three arrows, one from 1 to 1, one from 2 to 2, one from 3 to 3.

b) $S = \{(1, 3), (2, 1)\}$: Just reading off from the relation, the digraph has two arrows, one going from 1 to 3, one going from 2 to 1.

c) $\leq$: The elements of the relation $\leq$ will be pairs (a, b) such that $a \leq b$. The order here is the important thing. Since $1 \leq 3$, there will be an arrow *from* 1 *to* 3. The full relation is given by the elements $(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$. Don't forget that any number is less than or equal to itself.

d) S^{-1}, where $S = \{(1, 3), (2, 1)\}$: Taking the inverse of a relation just reverses the ordering, so $S^{-1} = \{(3, 1), (1, 2)\}$. This will produce a digraph identical to **b)**, but with the directions of the arrows reversed.

e) $\neq$: Elements will be of the form (a, b), where $a \neq b$. A number is not equal to every number except itself, so the arrows are $(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)$. You could either draw 6 arrows, or three bidirectional arrows, one between 1 and 2, one between 2 and

3, and one between 3 and 1.

f) $S \circ S$, where $S = \{(1,3),(2,1)\}$: To compose S with itself, start with the domain of S, which is 1 and 2. Applying S, those become 3 and 1, respectively. Applying S again, 3 is not in the domain of S, so it vanishes, but 1 is in the domain of S, and becomes 2. Hence, the only element of the relation $S \circ S$ is $(2,1)$. Therefore, the digraph will have one arrow, from 2 to 1.

Common Problems. Really just small errors here - arrows pointing the wrong direction, or arrows left out, etc.

Problem. 2: Let $A = \{a,b,c,d\}$. Give an example of relations R, S, T on A for each of the following.
a) $R \circ S \neq S \circ R$
b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$
c) $S \circ R = T \circ R$, but $S \neq T$
d) R and S are nonempty, but $S \circ R$ and $R \circ S$ are empty
1 Point each, 4 total.

Solution. a) $R \circ S \neq S \circ R$: Consider $R = \{(a,b)\}, S = \{(b,c)\}$. In that case, $S \circ R = \{(a,c)\}$, and $R \circ S = \emptyset$, since the range of S is not in the domain of R. Hence, $R \circ S \neq S \circ R$

b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$: Using the above example, $S \circ R = \{(a,c)\}$, so $(S \circ R)^{-1} = \{(c,a)\}$. However, $R^{-1} = \{(b,a)\}, S^{-1} = \{(c,b)\}$. Hence, $S^{-1} \circ R^{-1} = \emptyset$, again since the domain of S^{-1} has nothing in the range of R^{-1}. Hence, $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$.

c) $S \circ R = T \circ R$, but $S \neq T$: Take $R = \{(a,b)\}$. Let $S = \{(b,c),(c,d)\}, T = \{(b,c)\}$. Then taking $S \circ R$, we get that a is sent to b under R, then b is sent to c by S. Hence, $S \circ R = \{(a,c)\}$. Similarly, computing $T \circ R$, a is sent to b under R, then b is sent to c by T. Hence, $T \circ R = \{(a,c)\}$. I don't think anyone will argue when I say, $S \circ R = T \circ R$. And clearly, $S \neq T$.

d) R and S are nonempty, but $S \circ R$ and $R \circ S$ are empty : Consider $R = \{(a,b)\}, S = \{(c,d)\}$. Neither are empty, but composing R wth S or S with R will be empty, since the ranges and domains of R and S or S and R are disjoint.

Common Problems. Again, the main problems here were really small things. I'd say the most frequent error I saw was people composing backwards. When you compute $S \circ R$, you start with something in the domain of R, map it under R, and then map it under S. So if (a, b) is in R, and (b, c) is in S, you end up with (a, c) in $S \circ R$. The important thing to remember is that you **start** in R, and **end** in S.

Problem. 3: Let R be a relation from A to B, and S be a relation from B to C.
a) Prove that $Dom(S \circ R) \subset Dom(R)$.
b) Show by example that $Dom(S \circ R) = Dom(R)$ may be false.
c) Give an example to show which of the following statements may be false: $Rng(S) \subset Rng(S \circ R)$ or $Rng(S \circ R) \subset Rng(S)$
1 Point each, 3 total.

Solution. a) Suppose that $a \in Dom(S \circ R)$. Then by the definitio of domain, for some c, $(a, c) \in S \circ R$. By the definition of composition, there exists some b such that $(a, b) \in R, (b, c) \in S$. Since $(a, b) \in R$ for some b, we have that $a \in Dom(R)$. Therefore, $Dom(S \circ R) \subset Dom(R)$.

b) Let $R = \{(1, 2), (2, 3)\}, S = \{(2, 3)\}$. In that case, $S \circ R = \{(1, 3)\}$. Therefore, $Dom(R) = \{1, 2\}, Dom(S \circ R) = \{1\}$. Clearly, $Dom(S \circ R) \neq Dom(R)$.

c) Let $R = \{(1, 2), (2, 3)\}, S = \{(2, 3), (1, 2)\}$. Then $S \circ R = \{(1, 3)\}$. Therefore, $Rng(S) = \{2, 3\}$, and $Rng(S \circ R) = \{3\}$. Clearly, $Rng(S \circ R) \subset Rng(S)$, but it is **not** true that $Rng(S) \subset Rng(S \circ R)$.

Common Problems. For parts b), c), people did mostly all right, with the usual problems with composing backwards or incorrectly.

There was, however, a lot of variation in people's solutions to part a). The proof I've given here is sort of the tried-and-true method for proving one set is a subset of another set. A lot of responses argued something to the effect of *'An element in the domain of $S \circ R$ must be in the domain of R, while an element in the domain of R need not be in the domain of $S \circ R$'*, and that is true and well and good, but I would appreciate more of a justification of those kinds of statements.

It would be incorrect to argue in the following way. *$Dom(S \circ R)$ is a subset of A, and $Dom(R)$ is a subset of A, therefore $Dom(S \circ R) \subset Dom(R)$.* This would be equivalent to aruging, since the odd numbers are a subset of the natural numbers, and the even numbers are a subset of the natural numbers, the odd numbers are a subset of the even numbers. The key thing in this proof is to consider the structure of $S \circ R$ with respect to R.