LECTURE 18 EXCERCISE SOLUTIONS

Problem. 1: Suppose $S \subset \mathbb{Z}$ is bounded below and $S \neq \emptyset$, then S has a smallest element. (4 Points)

Solution. Assume, for sake of contradiction, that S has no smallest element. We will prove that S must in fact be empty.

Since S is bounded below, let k be a number such that $s \geq k$ for all $s \in S$. Note - S contains no numbers less than k.

This is an induction type argument. Note that k cannot be contained in S. If it were, then k would be the smallest element of S, since S contains nothing less than k. Since S has no smallest element, S cannot contain k.

Assume that S contains none of the numbers from k to n. I argue that S cannot contain $n+1$. If it did, then since S contains none of the numbers from k to n, and none of the numbers less than k, $n+1$ would necessarily be the smallest element of S. But that is a contradiction, so we must conclude that $n+1$ is not in S.

Hence, by induction, S contains no number greater than or equal to k. Since S also contains no number less than k, it is clear that S is empty. This is a contradiction.

Hence, if S is bounded below and non-empty, S must have a smallest element.

Common Problems. For this problem, and for many induction type problems, I got a lot of '(blank) is true, then keep repeating this to infinity' kind of arguments. That's the basic sketch of an induction argument, but you really need to make your logic rigorous and *prove* your results. Now, this is an interesting problem because it seems a sort of anti-induction argument, since you're disproving a claim for every number n, but it takes the exact same form as any other induction type argument.

A great many papers I received had practically identical answers for this problem, almost word for word, all containing various misstatements such as that 'we want to show that all numbers greater than k are in S'. This makes really no sense in the context of the problem, and indeed these people went on to show the exact opposite of that claim. Very odd.

For the next problems, let $n \in \mathbb{Z}, d \in \mathbb{N}, S = \{m \in \mathbb{Z} | dm > n\}$.

Problem. 2: Show that $S \neq \emptyset$. (4 Points)

Solution. As the hint says, we need to show that $|n| + 1 \in S$. Thus, from the definition of S, it suffices to show that $d(|n| + 1) > n$. Note two things, first that since $d \in \mathbb{N}$, we have that $d \geq 1$. Also, in general, we have that $|n| \geq n$. Since $|n|$ is non-negative, we can multiply through the first inequality to say $d|n| \geq |n|$. Note, this is equality if $n = 0$ or if $d = 1$, but in all cases it is certainly true. Combining that with the second inequality, we get that $d|n| \geq n$. Going back to the first inequality, we can say that $d|n| + d \geq n + d \geq n + 1 > n$. Hence, $d(|n| + 1) > n$, and $|n| + 1$ is in S.

Common Problems. Not too many issues with this problem. It's perfectly reasonable to break this problem into cases based on the sign of n, but I don't think it's necessary given that you know $|n| \geq n$. One thing that I saw some people doing in this problem was picking specific values for n and d, and using those - but it's really a general statement. You can't prove this kind of thing by example, as I often hear myself saying.

One thing that I did see a lot of was people not using the definition of S, and trying to prove in some kind of general way that S wasn't empty. But without assumptions, like defining what S is, there's no reason S couldn't be empty. Again, a lot of people making the same argument word for word. This was another instance of people defining some statement $P(n)$ about elements in S, and then trying to prove it for all n - and then making very strange comments about things like $P(n) + P(m)$. You can't add statements.

Another group of responses (several papers, all the same), was actually very clever, but wrong. These people wrote $S = \{x \in d\mathbb{Z} | x > n\}$, and then tried to use the Well Ordering Principle in some way, since S has the obvious lower bound of n. **However**, the key problem with this is that their S is not equivalent to the S given in the problem. Their S contains multiples of d, whereas the correct S contains the m that are multiplied by d. But, interesting thought.

Problem. 3: Prove that S is bounded below. (4 Points)

Solution. Suppose that $m \in S$. What you would like to show is that $m \geq -|n|$, according to the hint. So, assume not. In that case, $m < -|n|$. Therefore, $dm < -d|n|$. Since $m \in S$, we have that $n < dm$, so $n < -d|n|$. Note then, by cases, that if $n = 0$, the inequality is false (since $0 = 0$). If $n > 0$, then $n = |n|$, and the inequality becomes $n < -dn$, which is false, since n is positive and $-dn$ must be negative. If $n < 0$, then let $k = -n$ and $k = |n|$, so k is positive. Then the inequality becomes $-k < -dk$, or $k > dk$. Dividing by k (k is non-zero), we get $1 > d$, which is a contradiction. Hence, in any case, assuming that $m < -|n|$ gives a contradiction. Hence, $m \geq -|n|$ if $m \in S$, so $-|n|$ is a lower bound for S.

Common Problems. A lot of people made mistakes on this one. But a lot of people were making the same mistakes, since their answers were almost identical, word for word. The most common mistake was that people would say, we know that $|n| + 1$ is in S, and then they would prove that $|n| + 1 \geq -|n|$, therefore $-|n|$ is a lower bound for S. This is wrong. All this shows is that $|n| + 1$ is bound by $-|n|$. You need to show that *every element* of S is bound, not just one.

Another common approach was to argue that because $-|n| \leq n$, and $n < dm$ for each $m \in S$, we have $-|n| < dm$ for all m in S, and is therefore a lower bound. This is problematic as well. It provides a lower bound, yes, but it provides a lower bound on dm, not on m. You could just as easily (and some did) argue that n is a lower bound for the same reason, but it would also be wrong. Consider the case where $n = 5$ and $m = 2$. We have that $n < 3 * m$, so m is in S, but it is not true that n is a lower bound for S. You need to bound the elements of S, and those are not dm.

One thing that I let pass, and I probably shouldn't have, was to argue that since $dm > n$, we have that $m > n/d$, so n/d is a lower bound on S. This is true, if you accept some assumptions about division that haven't really been discussed at this point, and are okay juggling between rational numbers and natural numbers. A much better approach would be one like I outline above, since it deals completely with natural number relations that have been well established by this point.

Problem. 4: Show that if $d(q+1) > n \geq dq$ and $d(p+1) > n \geq dp$, then $p = q$. (4 Points)

Solution. Assume that $p > q$. In that case, $p \geq q + 1$. Then, $dp \geq d(q + 1)$. Hence, since $n \geq dp$ and $dp \geq d(q + 1)$, $n \geq d(q + 1)$. This contradicts what we are assuming.

Similarly, if $p < q$ (and this is just the mirror image of the previous proof), $q \geq p + 1$, and you derive the same contradiction that $d(p + 1) > n \geq d(p + 1)$.

Common Problems. People were mostly all right on this problem. One very troubling thing was that some number of people, in considering the case that $p > q$, said, 'let $p = q+1$'. That's fine for a test case or an example, but yet again it is insufficient for the entire proof, since there are many numbers greater than q that are not $q + 1$. One way to salvage this would be to say, if it fails for $p = q + 1$, it must fail for any larger p as well, and justify that, but people did not do that.

The one other thing worth mentioning is that Theorem 8 gives a value for q that satisfies the above inequality. It is incorrect, however, to say 'let $q = l-1$, and then let $p = l-1$', because then you are assuming what you want to prove - that p and q have the same value. As always, work from what you know.