LECTURE 15 EXCERCISE SOLUTIONS

Problem. 1: Show that if $\{\{a\}\} = \{\{a\}, \{a, d\}\}$, then $a = d$. (4 Points)

Solution. We know that every element on the right is contained in the left. In particular, $\{a, d\} \in \{\{a\}\}$. Therefore, $\{a, d\} = \{a\}$. Again, as we have two equivalent sets, we know that every element of one is contained in the other. Therefore $d \in \{a\}$. Therefore, $d = a$.

Common Problems. There was one very common very wrong approach to this problem, that went something like this:

By the definition of set equality, we know that $\{a\} \in \{\{a\}, \{a, d\}\}$. Therefore, either $\{a\} = \{a\}$ or $\{a\} = \{a, d\}$. In the first case, we get that $a = a$. In the second case, (by similar argument as my solution above), we get $a = d$. Done.

Note, the above basically says the following on the following: *Since* $\{a\} \in \{\{a\}, \{a, d\}\}$ *is true,* $a = a$ *or* $a = d$ *is true.* Now, for an **or** statement to be true, one or the other of the propositions must be true - but you have no idea which. Adding to your troubles, $a = a$ is *always* true. So taking this line of approach, you have no way of knowing that $a = d$ is true, and indeed, it doesn't even need to be true for the or statement to be satisfied. So you need more information, or you need a different approach.

Another thing that caught my eye is that a number of people tried to proceed by assuming that $a = a$ is false, and working out some conclusion from there. Unfortunately, $a = a$ is always true. And if you assume a true thing to be false, you can actually prove anything you like. This is what is known in the trade as 'breaking math'. Don't do that.

Problem. 2: Use a truth table to verify $(P \wedge Q) \vee (P \wedge R) \iff P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (R \wedge S) \implies (P \vee R) \wedge (Q \vee S)$. (4 points)

Solution. Just the usual truth table nonsense. In the first case, note that both sides are false if P is false. If P is true, then both sides are equivalent to $Q \wedge R$. In the second case, recall that an implication is only false if the left side is true while the right side is false. If $(P \vee R) \wedge (Q \vee S)$ is false, then P and R are both false, or Q and S are both false. In either case, $(P \wedge Q)$ is false, and $(R \wedge S)$ is false. So, in the case where the right side is false, the left side is necessarily false.

Common Problems. Not a lot of issues here. The most common thing people lost credit for were missing rows in your tables (3 variable truth table has 8 rows, 4 variable truth table has 16!), or failing to either verify the iff or the implication (only computing the necessary columns for the subpropositions). There is a small fraction of you, however, who still have trouble with computing $\wedge$ and $\vee$ results. FALSE and FALSE is not TRUE.

Problem. 3: Prove that $(A \times B) \cap (A \times C) = A \times (B \cap C)$, and that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. (4 Points, 2 Each)

Solution. Recall that we can prove two sets equal by proving an element is in one iff that element is in the other. Therefore,

$$\begin{gathered}
(x, y) \in (A \times B) \cap (A \times C) \\
\iff (x, y) \in A \times B \wedge (x, y) \in A \times C \\
\iff (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\
\iff (x \in A) \wedge (y \in B \wedge y \in C) \\
\iff (x \in A) \wedge (y \in B \cap C) \\
\iff (x, y) \in A \times (B \cap C)
\end{gathered}$$

$$\begin{gathered}
(x, y) \in (A \times B) \cap (C \times D) \\
\iff (x, y) \in (A \times B) \wedge (x, y) \in (C \times D) \\
\iff (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) \\
\iff (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\
\iff (x \in A \cap C) \wedge (y \in B \cap D) \\
\iff (x, y) \in (A \cap C) \times (B \cap D)
\end{gathered}$$

Note the use of the equivalences (necessary for problem 4), that $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge (P \wedge R)$. And in the second proof, that $(P \wedge Q) \wedge (R \wedge S) \iff (P \wedge R) \wedge (Q \wedge S)$.

Common Problems. There weren't many common errors here. Some people didn't use the right equivalences. Some people didn't make sure their proofs ran in both directions (in other words, they proved set containment, but not equality). Note too, that in many ways, the 'set'-based statement we're proving looks very similar to the 'logic'-based statement we end up using. It is not enough, however, to say that because they are similar, that they are both true. You must -prove- set equality, using something like what I've sketched out here.

Problem. 4: Verify the equivalences used in problem 3. Namely, $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge (P \wedge R)$ and $(P \wedge Q) \wedge (R \wedge S) \iff (P \wedge R) \wedge (Q \wedge S)$. (4 Points)

Solution. Again, the usual truth table nonsense. You can see by inspection, however, that if any one of P, Q, R, S are false, then each side of both iffs are false. And if all of P, Q, R, S are true, then each side of both iffs are true. Thus, in any case, both sides of both iffs are always the same. Thus the equivalences hold.

Common Problems. No serious issues here.