LECTURE 10 EXCERCISE SOLUTIONS

Problem. 1: Given m letters distributed into n pigeonholes, with $m\,n$, show there exists a pigeonhole with two or more letters.

Solution. Following the hint, we let the universe be the set of pigeonholes, and $P(x)$ be the property that pigeonhole x contains 2 or more letters. Note then, that $\sim P(x)$ is the property that x contains 1 or 0 letters. We are therefore asked to prove $\exists x P(x)$. Proceeding as suggested in remark 3, we consider the opposite (the denial) of what we *want* to prove, and hope to arrive at a contradiction.

Assume the denial is true, $\sim \exists x P(x)$. This is equivalent to $\forall x(\sim P(x))$. What does this mean? This statement says that for each pigeonhole, there are either 0 or 1 letters in that pigeonhole. Noting that each letter is in some pigeonhole, we therefore conclude that there is at most one letter per pigeonhole, or at most m letters. Hence, $m \leq n$. However, by assumption we had that $m\,n$. Since this is a contradiction, we conclude that our original assumption $\sim \exists x P(x)$ is false.

If $\sim \exists x P(x)$ is false, then $\exists x P(x)$ is true. Therefore, there exists some pigeonhole with 2 or more letters in it.

Problem. 2: If a, b, c, d are natural numbers with $\frac{a}{b} < \frac{c}{d}$, then there is a natural number n such that $\frac{a}{b} < \frac{n}{b+d} < \frac{c}{d}$.

Solution. As has been stated, proving existence statements are really the only time you should consider proving by example. In this case, consider $n = a + c$. But that number is sort of useless by itself, so I'll describe my logic/computations. In some sense, the solution $a + c$ is suggested by the structure of the problem (just in a sort of visual way), but if you want something more precise, here's how I thought about it. In these sorts of problems, it's often important to keep in mind what we're given, so we'll start there. Rewriting $\frac{a}{b} < \frac{c}{d}$, we have that

$$ad < cb$$

Keep that in mind.
After that, I started by putting the inequality we're interested in in terms of a common denominator.

$$\frac{ad(b+d)}{bd(b+d)} < \frac{nbd}{(b+d)bd} < \frac{cb(b+d)}{db(b+d)}$$

Once you have everything in a common denominator, it's clear that what we're really after is an n such that

$$ad(b+d) < nbd < cb(b+d)$$

Or, just to be clear, if not suggestive,

$$(ad)d + (ad)b < nbd < (cb)d + (cb)b$$

The problem here is the $d * d$ and the $b * b$. We'd like to get something that is entirely dividible by bd, and hence is some integer multiple of bd. So, to relate $d * d$ and $b * b$ to bd, I went back to the given and wrote down the following.

$$(ad)d < (cb)d$$
$$(ad)b < (cb)b$$

Then we can say that, adding adb to each side of the first and cbd to each side of the second,

$$(ad)d + (ad)b < (cb)d + (ad)b$$
$$(cb)d + (ad)b < (cb)d + (cb)b$$

Thus we see that $(ad)d + (ad)b < cbd + abd < (cb)d + (cb)b$. Simplifying in the middle,

$$(ad)d + (ad)b < (a+c)bd < (cb)d + (cb)b$$

Since the above inequality is true, we may take $n = a + c$, and satisfy everything.

Common Problems. As I'm writing this, I have not graded the problems yet, but there is an important point to be made here. It is insufficient to simply write down $n = a + b$ as your answer. The problem asks you to **prove** that such a satisfying n exists, and as such it is insufficient to give an n without showing that it does satisfy the inequality - that's the proof part of it.

Problem. 3: Continuing from problem 2, assume that $bc - ad = 1$. Prove that n is unique.

Solution. Recall that in the previous problem, you were asked to show that such an n exists. Having exhibited such an n, you are now asked to, given the assumptions, prove that there is a unique such n.

There are two approaches you could take here. If you assumed that you knew what the unique n was (for instance, if you thought that the n you found in problem 2 was unique), then you could show that the inequality failed for $n+1$ and thus failed for all larger numbers, and failed for $n - 1$ and thus failed for all smaller numbers. Therefore, it is only satisfied for your particular n, and it is unique. If you did not know what the unique n was, then what you could do is work out, given a, b, c, d, the range of n that satisfied the inequality.

For instance, for some N_{min}, N_{max}, if $N_{min} \leq n \leq N_{max}$, then n satisfies the inequality. You could then try to show that, if $bc - ad = 1$, then there is a single natural number between N_{min} and N_{max}, and prove uniqueness of n that way.

However, I believe that my answer of $n = a + c$ is the unique answer in this case. So I want to show that

$$\frac{a+c-1}{b+d} \leq \frac{a}{b}$$

$$\frac{c}{d} \leq \frac{a+c+1}{b+d}$$

The above are equivalent to

$$b(a+c-1) \leq a(b+d)$$
$$c(b+d) \leq d(a+c+1)$$

A little algebra...

$$ba + bc - b \leq ab + ad$$
$$cb + cd \leq da + dc + d$$

Cancel stuff...

$$bc - b \leq ad$$
$$cb \leq da + d$$

Rearrange in a suggestive way...

$$bc - ad \leq b$$
$$bc - ad \leq d$$

Now, to review, if the above inequalities are true, then it is true that $\frac{a+c-1}{b+d} \leq \frac{a}{b}$ and $\frac{c}{d} \leq \frac{a+c+1}{b+d}$ are true. If those are true, then $n = a + c$ is the unique n that satisfies the inequality in problem 2.

So it suffices to show that $bc - ad \leq b$ and $bc - ad \leq d$. However, by assumption, $bc - ad = 1$. Thus these inequalities are equivalent to showing $1 \leq b$ and $1 \leq d$. Since b and d are natural numbers, these inequalities are true.

Therefore, $bc - ad \leq b$ and $bc - ad \leq d$ are true.

Therefore, $\frac{a+c-1}{b+d} \leq \frac{a}{b}$ and $\frac{c}{d} \leq \frac{a+c+1}{b+d}$ are true.

Therefore, $n = a + c$ is the unique n such that the inequality in problem 2 is satisfied.