LECTURE 1 EXCERCISE SOLUTIONS

Problem. 1: Show using a truth table that $(\sim P) \vee (\sim Q)$ is a denial of $P \wedge Q$.

Solution. What does it mean to be a denial? A denial of a statement S (where S may be a function of several variables) is equivalent to $\sim S$.

In this case here, note that $(\sim P) \vee (\sim Q)$ is false only when both P and Q are true. In any other case, $(\sim P) \vee (\sim Q)$ will be true.

Similarly, note that $P \wedge Q$ is -true- only when both P and Q are true. In any other case, $P \wedge Q$ will be -false-.

Hence we see that, no matter what the values of P and Q are, the two statements are always the opposite of each other.

Common Problems. If I took points off on this problem, it was generally just for some computational error in your truth table. You really should know the truth tables for $\sim$, $\wedge$, $\vee$ by heart.

Problem. 2: Show using a truth table that $(P \wedge Q) \vee R$ is not equivalent to $P \wedge (Q \vee R)$.

Solution. The problem asks for a truth table, but that is just checking every single possible case. Since you want to show that they are not equivalent, it suffices to find one way of setting P, Q, R such that the two statements are not equal.

Consider, for instance, the case when P is false and R is true.

In that case, $P \wedge Q$ is necessarily false. But R is true, so $(P \wedge Q) \vee R$ is true.

Similarly, '$P\wedge$ **anything**' is necessarily false. Hence, $P \wedge (Q \vee R)$ is false

Since, when P is false and R is true, the two statements are opposite, the two statements over all are not equivalent.

Common Problems. Again, if there were problems here, they were generally again just computational errors in working out the different columns of your table. However, one other thing that I took of points for was people not showing their work. In a case like this, where you have moderately complex statements (and especially when they would get much more complicated), I don't think it is sufficient to write out the columns for P, Q, R, and then write out directly the columns for $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$, without doing the columns for the sub-expressions as well. If you just write out the final columns and they are correct, I have no way of knowing that you didn't just copy them from a friend or the internet. If you just write out the final columns and they are wrong, I have no way of knowing what you did wrong. Partial credit (for when I can see what you did) is better than no credit (when I assume you didn't do anything).

Problem. 3: Xor, or $\oplus$, is defined so that $P \oplus Q$ is true if and only if exactly -one- of P, Q is true. Show using a truth table that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \sim (P \wedge Q)$.

Solution. For $(P \vee Q) \wedge \sim (P \wedge Q)$ to be true, $P \vee Q$ must be true and $P \wedge Q$ must be false.

For $P \vee Q$ to be true, one or both of P, Q must be true.

For $P \wedge Q$ to be false, one or both of P, Q must be false.

The only way for both of these to occur is if one of P, Q is true and the other is false.

In any other case, $(P \vee Q) \wedge \sim (P \wedge Q)$ must be false.

Hence, $(P \vee Q) \wedge \sim (P \wedge Q)$ is true if one of P, Q is true, and false otherwise. Thus it is equivalent to $P \oplus Q$.

Common Problems. Again, problems here were largely limited to computational errors in filling in your columns, or not showing sufficient work.

Problem. 4: Let $f(P, Q)$ and $g(P, R)$ be two tautologies; show that f, g are equivalent by considering a simultaneous truth table in P, Q, R. Are any two tautologies equivalent, *even if they involve different variables*?

Solution. Recall that a tautology is a statement that is always true, regardless of the values of its variables. Consider any values you like for P, Q, R. No matter what you chose for P, Q, $f(P, Q)$ is true. Similarly, no matter what you chose for P, R, $g(P, R)$ is true. This shows that on every line of your truth table, you will have f and g both equal to true, and thus they are equivalent.

Any two tautologies are always equivalent, regardless of the variables. For one thing, variables are just names. If I have a numerical function $f(x) = 4$ and another $g(y) = 4$, the fact that x and y are different letters doesn't change the fact that f and g depend on their variables *in the same way.*

Another way of thinking about it: If you like, in this case, you could invent new statements we'll call $h(P,Q,R)$ and $j(P,Q,R)$ defined so that

$$h(P,Q,R) = f(P,Q)$$
$$j(P,Q,R) = g(P,R)$$

In this case, it is clear that h and j are still tautologies, and thus always true. Thus they are equivalent. And since $h = f$ and $j = g$, we get that $f = g$. Note that h is a function of R, but does not -depend- on R. j is a function of Q, but does not -depend- on Q.

Common Problems. I got a lot of answers on this. The main thing that people said that I took issue to is that 'f and g cannot be equivalent, since they depend on different variables'. But as I said, there is some subtlety regarding variables, the names of variables, and dependence. I don't really recall what other errors people made, but when you are told that something is a 'tautology', you should know that its column in a truth table is going to be all true.