RESEARCH STATEMENT

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I am interested in the intersection of model theory, which is a branch of mathematical logic, with other fields of mathematics. Both classical and modern model theory have found numerous applications to algebra, diophantine problems, and more recently to a variety of combinatorial questions. To date, I have been studying algebraic and dynamical properties of the automorphism groups of metrically homogeneous graphs, which are objects that lie at the interface of model theory and combinatorics. My expectation for the near future is to consider some problems which are natural outgrowths of my current work as well as other aspects of model theoretic combinatorics and model theoretic algebra.

Results

Background: My results concern metrically homogeneous graphs, which are graphs which, when endowed with the path metric, are homogeneous in the sense that every partial isometry can be extended to a full isometry. A conjectured classification is given by Cherlin in [5].

Homogeneity is a compelling concept intrinsically, and recent work has shown homogeneous objects to be a bridge between diverse areas of mathematics. In 2005, Kechris, Pestov, and Todorčević published their seminal paper *Fraïssé Limits, Ramsey Theory, and Topological Dynamics of Automorphism Groups* [17] which brought to light connections between topics in model theory, combinatorics, and descriptive set theory. Also in 2005, Nešetřil detailed some of these ties in his paper *Ramsey Classes and Homogeneous Structures* [19]. These results make a compelling case for finding more and different examples of homogeneous structures. Moreover, the study of homogeneous structures provides a fruitful introduction to many areas of research.

Countable metrically homogeneous graphs are similar to both the Random Graph as well as to rational Urysohn space, which are themselves also countable homogeneous structures. We may therefore take questions asked about the Random Graph and Urysohn space and apply them to these metrically homogeneous graphs. Or from another perspective, we may ask: what makes the Random Graph or Urysohn space different from countable homogeneous structures?

Classification of Twisted Automorphisms

We introduce the concept of *twisted isomorphisms*, which are isomorphisms up to a permutation — or *twist* — of the language. For example, a non-trivial twist of graphs within the language $\{E, E^c\}$, that is, where edge and non-edge each have a relation, would result in a map switching edges and non-edges.

Twisted isomorphism is a very natural notion of similarity, and analogous concepts have been studied separately by Herwig, Ivanov, and Bagheri, under different names. We first explored this concept in relation to the classification of metrically homogeneous graphs of diameter 3 in [6], and a classification of twists became helpful in simplifying the analysis there.

The natural language of a graph of bounded integral diameter δ, when viewed as a metric space via the path metric, is $\{R_1, R_2, ..., R_\delta\}$ where each R_i is a binary relation such that $R_i(x, y)$ if and only if $d(x, y) = i$, and where d is the path metric. In this context, a twist would be a permutation of $\{R_i\}_{i=1}^{\delta}$ which maps some metrically homogeneous graphs to some metrically homogeneous graphs. We note that even preserving metric spaces is a task, since the triangle inequality will likely be violated by a randomly chosen permutation.

In [13] we classify all the twists of metrically homogeneous graphs, despite the classification of metrically homogeneous graphs themselves being unconfirmed. We also classify all metrically homogeneous graphs which are twistable, i.e. graphs which permit a twist, in terms of the numerical parameters described below.

Surprisingly, there are only four non-trivial twists of a large, well-behaved, and unified class of metrically homogeneous graphs said to be of "generic type", and these twists have nice format. We show them here:

Theorem. *Let σ be a non-trivial permutation of the language of a metrically homogeneous graph Γ of generic type where Γ^σ is itself a metrically homogeneous graph. Then σ is one of $\rho, \rho^{-1}, \tau_0, \tau_1$, which are defined as follows:*

$$\rho(i) = \begin{cases} 2i & i \leq \delta/2 \\ 2(\delta - i) + 1 & i > \delta/2 \end{cases} \qquad \rho^{-1}(i) = \begin{cases} i/2 & i \text{ even} \\ \delta - \frac{i-1}{2} & i \text{ odd} \end{cases}$$

and for $\epsilon = 0$ or 1

$$\tau_\epsilon(i) = \begin{cases} (\delta + \epsilon) - i & \text{for } \min(i, (\delta + \epsilon) - i) \text{ odd} \\ i & \text{otherwise} \end{cases}$$

The classification of twists of metrically homogeneous graphs proved useful not only in the classification of metrically homogeneous graphs of diameter 3, given in [6], but also in my analysis of the finiteness problem for metrically homogeneous graphs, described below in *Ramsey theoretic and dynamical properties of metrically homogeneous graphs.* My work in twists also led to new results on the structure theory for metrically homogeneous graphs with some relevance to the classification program.

There is a five-parameter family of constraints $(\delta, K_1, K_2, C, C')$ which, along with an additional parameter $\mathcal{S}$, determine the known, in the sense that they are in Cherlin's catalog, metrically homogeneous graphs of generic type. I found the necessary and sufficient conditions on $(\delta, K_1, K_2, C, C')$ such that metrically homogeneous graphs satisfying these parameters permit a non-trivial twist. I also found all twistable metrically homogeneous graphs of non-generic type; the classification of metrically homogeneous graphs of non-generic type is known to be complete.

The final version of [13] will be submitted for consideration in a planned special issue of the *European Journal of Combinatorics.*

Future Work: Bannai and Bannai examined in [2] a similar concept to twists, applied to association schemes. There is a direct link between association schemes and distance regular graphs–for more on this connection, see [20]. Thus we may view their results as classifying a concept similar to twists for finite distance regular graphs. Strikingly, they find twists which are identical to the ones found for metrically homogeneous graphs of generic type. A natural follow-up would therefore be to find a common generalization of these two contexts. A natural such generalization would be to the classification of the twists of distance transitive graphs; a census of the then known distance transitive graphs is given in 1998 by Cameron in [4].

Splitting of the Twisted Automorphism Group

We may examine the group of twisted automorphisms as an extension of the standard automorphism group. Cameron and Tarzi considered two problems concerning the twisted automorphism group of the m-edge-colored random graph; we call these problems the *splitting* and the problem of *strong permutation completeness*. We consider the analogous problems for metrically homogeneous graphs. The splitting problem asks when the group of twisted automorphisms splits over the automorphism group. The strong permutation completeness problem asks when the twisted automorphism group of metrically homogeneous graph taken modulo the center of the automorphism group of the metrically homogeneous graph can be identified with the full automorphism group of the group of automorphisms of this metrically homogeneous graph. In other words, when does the action of the twisted automorphism group $Aut^*(\Gamma)$ on $Aut(\Gamma)$ induce $Aut(Aut(\Gamma))$?

An answer to the splitting question — namely that when the twisted automorphism group differs from the automorphism group, we would expect the extension to split, and when it does not there are specific obstructions — is given in joint work with Gregory Cherlin in [7]. Once again the generic case is distinguished from the non-generic case. One result of splitting in the generic case is as follows:

Theorem. *Let Γ be a known metrically homogeneous graph of generic type and diameter $\delta \geq 3$ which is twistable by the involution τ_ϵ.*

Then $Aut^*(\Gamma)$ splits over $Aut(\Gamma)$ if and only if

$$\delta + \epsilon \not\equiv 3 \mod 4.$$

We found some natural obstructions to splitting, including one of a new kind involving central automorphisms. In cases where splitting was proved, we needed to build automorphisms, and this was typically done by a construction that builds the automorphisms and the structure on which they act simultaneously.

The question of permutation completeness is the subject of ongoing work. Whether the automorphism group of the automorphism group is induced by the twisted automorphism group is related to issues of automatic continuity which in the context of homogeneous structures are often resolved by recourse to the small index property, which intriguingly is one of the dynamical properties discussed in the next section. We believe that for a metrically homogeneous graph Γ the group $Aut^*(\Gamma)/Z(Aut(\Gamma))$ will induce the group of continuous automorphisms of $Aut(\Gamma)$, apart from certain specific cases.

Future Work: Cameron and Tarzi's m-edge-colored random graph and our metrically homogeneous graphs have a natural common generalization to homogeneous structures in symmetric languages. Thus, we ask: for which homogeneous structures in symmetric languages does the group of twisted automorphisms split over the standard automorphism group? For which homogeneous structures Γ in a finite symmetric relational language does the action of the twisted automorphism group of Γ on $Aut(\Gamma)$ induce all continuous automorphisms of $Aut(\Gamma)$? One type of homogeneous structure in a symmetric languages which we may consider is the generic m-edge-colored k-hypergraph. In the *binary* symmetric case, the results to date suggest that splitting may be closely connected with the behavior of the fixed points of involutive twists, but there is no general conjecture as to what form this criterion should take.

Ramsey theoretic and dynamical properties of metrically homogeneous graphs

As shown in Kechris-Pestov-Todorčević [17] and in Kechris-Rosendal [18], the automorphism groups of certain homogeneous structures can have quite nice topological dynamical properties. Work of Herwig-Lascar [15] combined with results from Hubička-Nešetřil [16] yield the combinatorial conditions which are "usually" (this word is actually used by Kechris and Rosendal in [18], found on page 32) sufficient to apply the results of Kechris-Pestov-Todorčević and Kechris-Rosendal. Specifically, a finiteness condition and a completion procedure applied to these homogeneous structures is sufficient to establish these topological dynamical properties.

The required finiteness condition in the context of metrically homogeneous graphs is as follows. We consider a class $\mathcal{A}$ of all (up to isomorphism) finite induced subgraphs of a given metrically homogeneous graph; these are certain finite metric structures. We then consider *partial* $\mathcal{A}$-structures. These are in particular partial metric spaces; since we view metric spaces as complete edge-colored graphs, there are possibly incomplete edge-colored graphs which have completions in the class $\mathcal{A}$. We require that the partial $\mathcal{A}$-structures should be characterized by finitely many forbidden partial structures. In fact, we require more: we require that there be a canonical completion process which completes any partial structure which can be completed in the class $\mathcal{A}$, and whose analysis gives the finite set of possible obstructions.

In [12] I show that these classes are in fact finitely constrained. Once I proved that amalgamation classes of metrically homogeneous graphs are finitely constrained, then I was able to show, using results from Herwig-Lascar, that these classes satisfy the *Hrushovski property*. Afterwards, I used results from Kechris and Rosendal to show that these Fraïssé limits have ample generics. From there we have that the automorphism groups have a whole host of nice topological dynamical properties, e.g. the small index property, uncountable cofinality, automatic continuity, and finally, using results from Hubička-Nešetřil [16], that the universal minimal flow is metrizable.

One clean result distills to the following.

Theorem. *Let Γ be a metrically homogeneous graph of generic type, which is non-bipartite and which satisfies $C > 2\delta + \min(K_1, \delta/2)$ and $K_2 \geq \delta - 1$ (note that C trivially is at least $2\delta + 1$).*

Then $Aut(\Gamma)$ has ample generics and consequently has a variety of topological dynamical properties, including the small index property, uncountable cofinality, and automatic continuity.

Ramsey theoretic results from Hubička and Nešetřil in [16] also allow us to deduce that the universal minimal flow of $Aut(\Gamma)$ is metrizable.

More general results were recently obtained by Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatcher, Konečný, and Pawliuk in [1], whose work did not require a restriction on the parameters C, K_2.

Continuing and future work

I am interested in applications of model theory as well as in pure model theory. The above descriptions already mention some future work which are natural outgrowths of my thesis. Here I discuss some additional questions I would like to address.

(a) *Model theoretic classification of theories of metrically homogeneous graphs*

In 2014, Conant and Terry examined the model theoretic properties of the Urysohn sphere in [11]. In particular, they showed that the Urysohn sphere is SOP_n for all $n \geq 3$, but not SOP_∞. The known metrically homogeneous graphs of generic type are similar to Urysohn space in that they are both metric structures. Thus, we have reason to believe that these metrically homogeneous graphs are also SOP_n for large enough n. Conant and Terry studied the Urysohn sphere as a metric structure in continuous logic. Thus in order to use their techniques, we must ask what a continuous logic analog for metrically homogeneous graphs would be. Conant and Terry also showed that once they were able to examine the Urysohn sphere as a continuous structure, they could prove similar properties for corresponding discrete structures. Thus, constructing appropriate discrete analogs for metrically homogeneous graphs should allow us to prove similar model theoretic properties for metrically homogeneous graphs as structures in first-order logic. This would also allow us to explore the following: is there divergent behavior between classes of metrically homogeneous graphs of increasing diameter, which can be viewed, when normalized, as finer and finer approximations of a continuous metric space, and metrically homogeneous graphs defined in continuous logic?

Proposal: Study the questions considered in [11] in the context of metrically homogeneous graphs, relative to classical logic (as the metric topology is discrete, and the diameter is sometimes infinite). Also develop a theory of continuous analogs of metrically homogeneous graphs – including Urysohn space, but looking for a broad range of examples similar to those known in the discrete case.

(b) *Algebra of an age*

In [3], Peter Cameron presented two methods for proving that the *algebra of an age* of an oligomorphic permutation group is a polynomial algebra, where the age of a structure is the class of all finite structures which embed into it as induced substructures. In the second method, Cameron requires a "*good notion of connected*" ([3], page 3), which is correspondingly axiomatized. Moreover, Cameron requires that the age $\mathcal{A}$ permit a compatible notion of a partial order, and that there is a natural notion of structure composition, satisfying certain constraints. The exact meaning of these terms can vary in context, but they all should be well-defined, without much difficulty, in the case of metrically homogeneous graphs. This is seen as there exists a natural correspondence between countable homogeneous relational structures and oligomorphic permutation groups, and as in the context of metrically homogeneous graphs there are likely candidates for the notions Cameron provides.

Proposal: Determine whether the conditions provided by Cameron are satisfied by known metrically homogeneous graphs of generic type and deduce that the algebras of the corresponding ages are polynomial algebras. Determine generally which metrically homogeneous graphs have polynomial age algebra.

(c) *Forbidden subgraphs*

The Erdős-Hajnal conjecture is the following:

Conjecture. *For any graph H, there is a constant $c(H) > 0$ such that any graph G forbidding H must contain a clique or an independent set of size at least $|V(G)|^{c(H)}$.*

This is a famous conjecture in combinatorics and has proven notoriously difficult. The state of current knowledge is succinctly and comprehensibly described in a survey by Chudnovsky in [10]. Much attention recently has been given to the Erdős-Hajnal conjecture by model theorists. Most of the work to date has been attempting to prove the conjecture for classes of graphs which satisfy certain nice model theoretic properties. See [8], [9], and [14] for a partial list.

I would like to examine the reverse question, namely–to explore the model theoretic properties of the class of H-free graphs for those H for which the Erdős-Hajnal conjecture has been proved, as well as of similar classes which appear to be well-behaved structurally. For example, what are the model theoretic properties of classes of graphs with short forbidden induced subpaths?

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