Randomized Greedy Online Edge Coloring Succeeds for Dense and Randomly-Ordered Graphs

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[1]University of Salzburg

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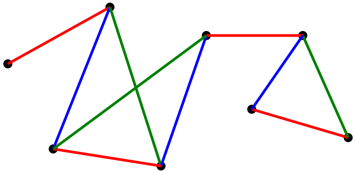
Edge Coloring

- Edge coloring: assignment of colors to edges of G so that no two edges adjacent to the same vertex have the same color.



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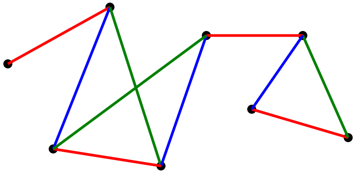

- Vizing: Graph with maximum degree Δ can be properly edge colored with $\Delta + 1$ colors (and requires at least Δ colors).

Online Algorithms

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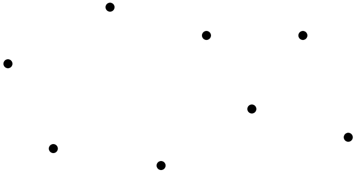

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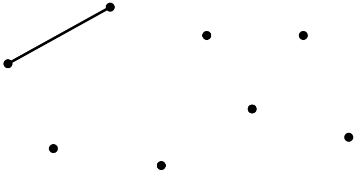

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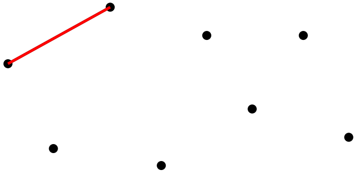

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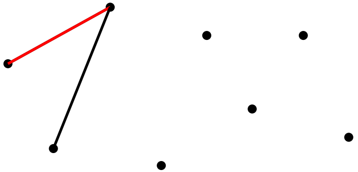

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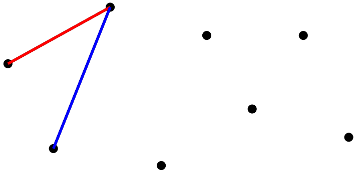

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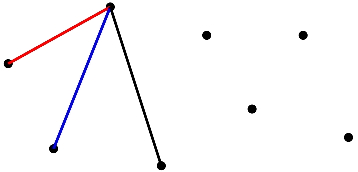

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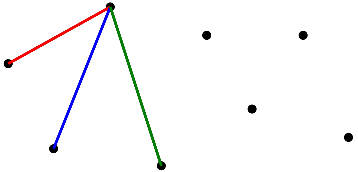

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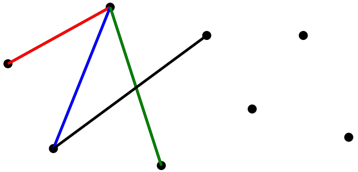

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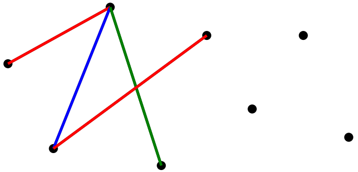

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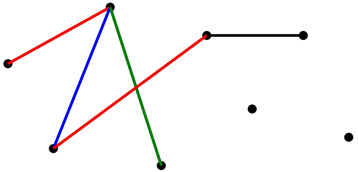

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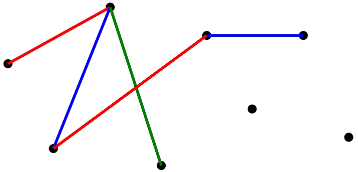

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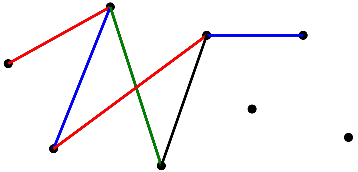

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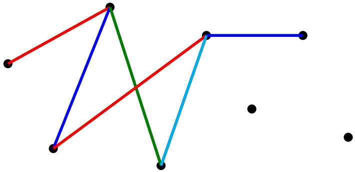

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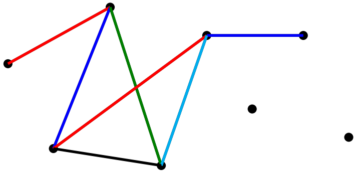

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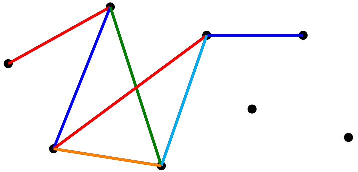

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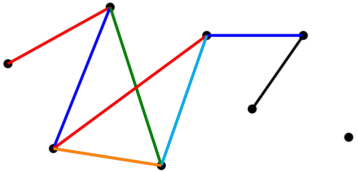

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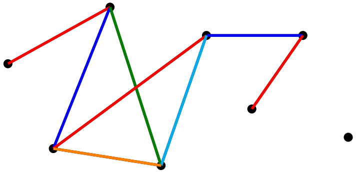

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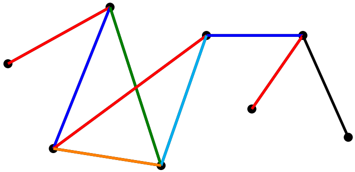

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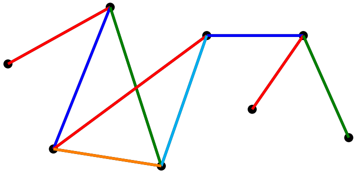

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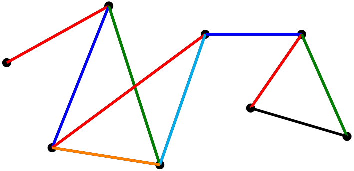

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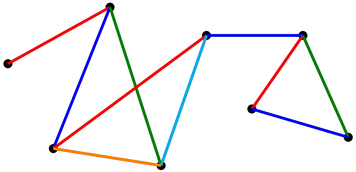

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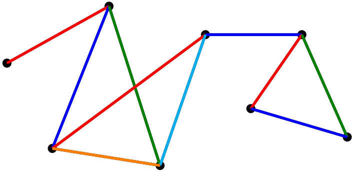

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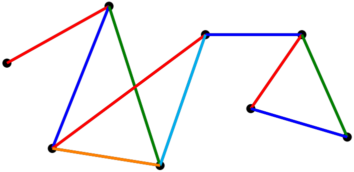

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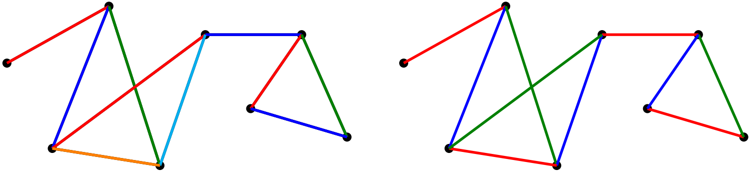

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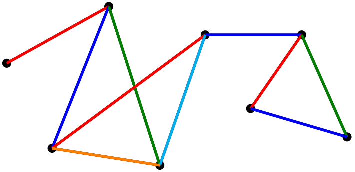

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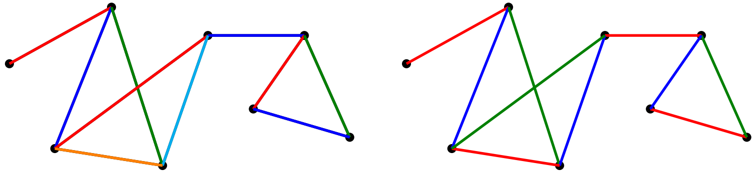

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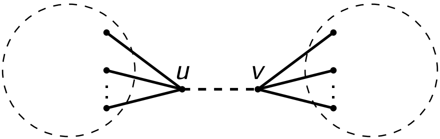
- Random greedy algorithm: start with a color set of size $(1 + \epsilon)\Delta$ and choose the color for each edge uniformly at random from the valid remaining colors upon arrival.

Intuition: Trees [Feder, Motwani, and Panigrahy n.d.]



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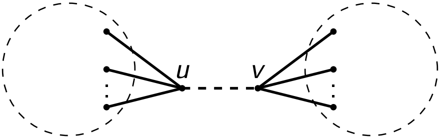

- Let $F(u)$ and $F(v)$ be the set of free (unused) colors from the palette at u and v respectively when edge (u, v) arrives.

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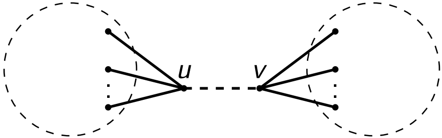

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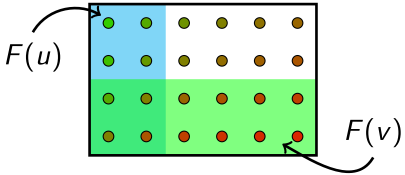
- We expect



$$|F(u) \cap F(v)| \approx \frac{|F(u)||F(v)|}{(1+\epsilon)\Delta} \geq \frac{\epsilon^2 \Delta}{1+\epsilon}.$$

Results

Adversarial Settings:

- Random Order Arrival
- Oblivious Adversary
- Adaptive Adversary

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Theorem (Random Order Case)

For all ϵ, there exists a constant N s.t. if the edges of a graph G with maximum degree Δ on $n \leq 2^{\frac{\Delta}{N}}$ vertices arrive in a random order, then with high probability, the random greedy algorithm using $(1+\epsilon)\Delta$ colors succeeds.

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Theorem (Dense Case)

For all ϵ, M and $n \leq M\Delta$, the random greedy algorithm using $(1+\epsilon)\Delta$ colors succeeds with high probability on graphs with maximum degree Δ on n vertices, even if the edges of G are chosen by an adaptive adversary.

Results

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Corollary (Deterministic Case)

For all ϵ, M there is a deterministic algorithm that, for Δ sufficiently large, will $(1+\epsilon)\Delta$ color graphs with maximum degree Δ on $n \leq M\Delta$ vertices.

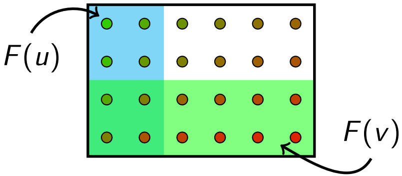
Related Work

- Bar-Noy, Motwani, and Naor 1992: No random online algorithm using $< 2\Delta - 1$ colors can guarantee success with probability $\geq \frac{1}{e}$.
- Bhattacharya, Grandoni, and Wajc 2021: Online $(1 + o(1))\Delta$ coloring algorithm that succeeds whp in random order $\Delta = \omega(\log n)$ setting.
- Blikstad et al. 2024b: Provides an online $(1 + o(1))\Delta$ coloring algorithm that succeeds with high probability when $\Delta = \omega(\log n)$.
- Blikstad et al. 2024a: Provides a deterministic $(\frac{e}{e-1} + o(1))$-competitive online bipartite edge-coloring algorithm under one-sided vertex arrivals when $\Delta = \omega(\log n)$.

Key Idea

- Intuition from tree case: for all pairs u, v, we want the sets $F(u)$, $F(v)$ to look independent. That is, we would like

$$|F(u) \cap F(v)| \approx \frac{|F(u)||F(v)|}{(1+\epsilon)\Delta}$$



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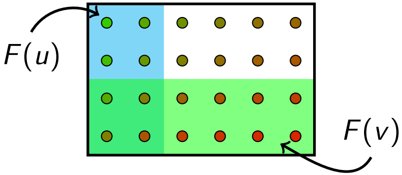

- It suffices if at each step, the color c assigned to an edge of v satisfies

$$\Pr[c \in F(u)] \approx \frac{|F(u) \cap F(v)|}{|F(v)|}$$

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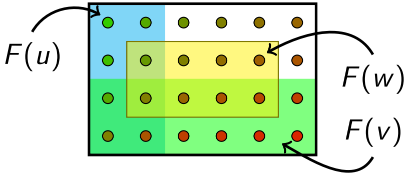
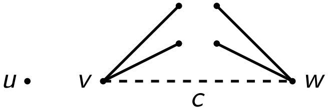


- Reality:



$$\Pr[c \in F(u)] = \frac{|F(u) \cap F(w) \cap F(v)|}{|F(v) \cap F(w)|}$$

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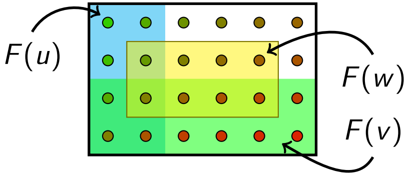

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- Note: it is enough to show that for all colors sets S and vertices w,

$$\frac{|F(w) \cap S|}{|F(w)|} \approx \frac{|S|}{(1+\epsilon)\Delta}. \qquad (*)$$

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- What we can show: with high probability, for all S, all vertices w excepting constantly many satisfy $(*)$.

Tracking $|F(v) \cap S|$

$v \bullet$

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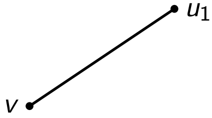

$\Pr[c_1 \in S] = \frac{|F_0(u_1) \cap F_0(v) \cap S|}{|F_0(u_1) \cap F_0(v)|} = \frac{|F_0(v) \cap S|}{|F_0(v)|} \pm \widehat{\epsilon}_1$

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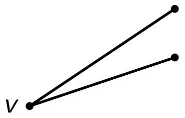

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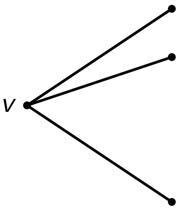

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$\vdots$

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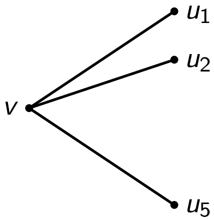

$$\Pr[c_1 \in S] = \frac{|F_0(v) \cap S|}{|F_0(v)|} \pm \widehat{\epsilon}_1$$

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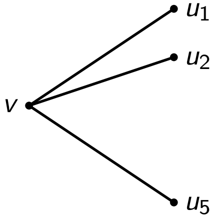
$$\vdots$$

$$\Pr[c_5 \in S] = \frac{|F_4(v) \cap S|}{|F_4(v)|} \pm \widehat{\epsilon}_5$$

- Let $X_i(S)$ indicate whether $c_i \in S$ and $p_i(S) = \Pr[c_i \in S]$

$$|S \cap \overline{F(v)}| = \sum X_i(S) \quad = \sum (X_i(S) - p_i(S)) + \sum \frac{|F_{i-1}(v) \cap S|}{|F_{i-1}(v)|} + \sum \widehat{\epsilon}_i$$

Tracking $|F(v) \cap S|$



$$\Pr[c_1 \in S] = \frac{|F_0(v) \cap S|}{|F_0(v)|} \pm \widehat{\epsilon}_1$$

$$\Pr[c_2 \in S] = \frac{|F_1(v) \cap S|}{|F_1(v)|} \pm \widehat{\epsilon}_2$$

$$\vdots$$

$$\Pr[c_5 \in S] = \frac{|F_4(v) \cap S|}{|F_4(v)|} \pm \widehat{\epsilon}_5$$

- Let $X_i(S)$ indicate whether $c_i \in S$ and $p_i(S) = \Pr[c_i \in S]$

$$|S \cap \overline{F(v)}| = \sum X_i(S) \quad = \sum (X_i(S) - p_i(S)) + \sum \frac{|F_{i-1}(v) \cap S|}{|F_{i-1}(v)|} + \sum \widehat{\epsilon}_i$$

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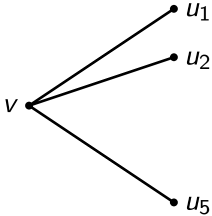

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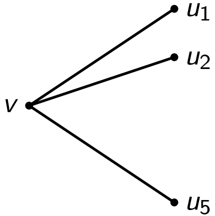

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- Two sources of divergence in our calculation of $|S \cap F(v)|$:
  1. The natural divergence between $X_i(S)$ and $p_i(S)$
  2. The $\widehat{\epsilon}$ error terms from each neighbor

Future Directions

- Generalizing the dense case:
    - Current proof of the error bound for the dense case allows for worst case directions of the errors of all neighbors.
    - Can we improve our estimate of the collective error of sets of vertices in order to weaken the density requirement?

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- Multigraphs:
  - In the case of multigraphs, instead of Vizing's theorem, we have Shannon's theorem giving an upper bound of $\frac{3\Delta}{2}$ for $\chi'(G)$.
  - How close can the random greedy algorithm get to this bound in the multigraph case?

Thank You

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