RYSP 2021: Graph Theory Notes

1 Basic Definitions

1.1 Intro

- Goals:
  1. Learn how to think about math
     - Learn how to approach a problem and begin thinking of solutions.
     - Speed doesn't matter and getting the "right" answer doesn't matter - the important thing is to spend time thinking and hopefully understand something more about the problem at the end.
     - There are many different ways to think about the same problem.
  2. Theorems and applications
     - Learn about the fundamental questions and results in various areas of graph theory.
     - See how different real-world situations can be understood through the framework of graph theory.
  3. Learn how to communicate math
     - Learn how to present the solution to a problem clearly so that other people can understand.
- Course Set-Up
  - Short discussion of material followed by time to try exercises and discuss in groups
  - Two types of exercises:
    1. To check understanding of a definition or question.
    2. To help figure out how to prove a result.
  - Homework session with additional problems and presentation of selected homework problems from the previous day.
    * A note about doing the homework: collaboration between students is highly encouraged; looking up solutions to problems is strongly discouraged. In any case, please clearly state the names of any collaborators or resources used to solve a problem when presenting the solution.
- References:
  1. *Discrete Mathematics with Ducks*, sarah-marie belcastro
  2. *Discrete Mathematics and Its Applications*, Kenneth H. Rosen

1.2 A History of Graph Theory

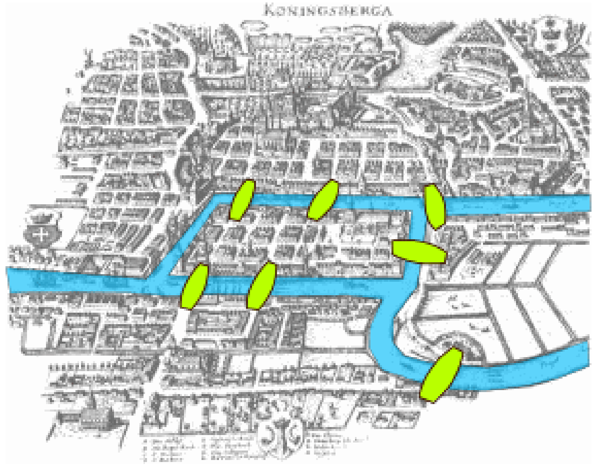


Figure 1: Seven Bridges of Königsberg (source: Wikipedia)

Exercise: Is there a way to walk through the city crossing every bridge exactly once if:

1. You want to start and end in the same part of the city?
2. You want to start and end in different parts of the city?

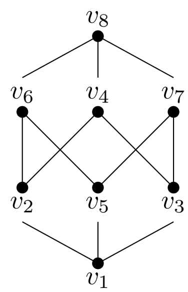
Discussion:

1. What information is relevant to answering this question?
2. How can you prove your answer is true?

1.3 Basic Definitions

- Formally, a (simple) graph $G = (V, E)$ where:
  - $V = V(G)$, the vertex set of G, is a nonempty, finite set of vertices, and
  - $E = E(G)$, the edge set of G, is a set of edges of the form $\{u, v\}$ for $u \neq v \in V$.

G :



$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

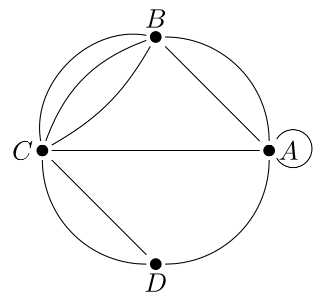
$$E(G) = \{\{v_1, v_2\}, \{v_1, v_5\}, \{v_1, v_3\}, \{v_2, v_6\}, \{v_2, v_4\}, \{v_5, v_6\}, \{v_5, v_7\}, \{v_3, v_4\}, \{v_3, v_7\}, \{v_6, v_8\}, \{v_4, v_8\}, \{v_7, v_8\}\}$$

Note: in this definition, there can only be one edge between any two vertices. This is different from the map of Königsberg shown above.

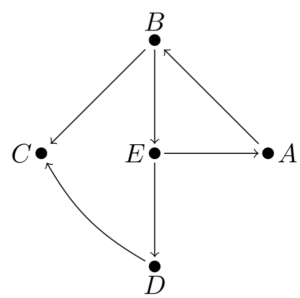
- Two vertices are *adjacent* if they are contained in an edge and that edge is *incident* to those vertices. Two edges are *incident* if they have a vertex in common.
- The *degree* of a vertex v, $d(v)$, is the number of edges incident to it.
- The *neighborhood*, $N(v)$ of a vertex v is the set of vertices adjacent to v.
- We denote the number of edges in graph G by $e(G)$ and the number of vertices by $v(G)$.
- The *degree sequence* of a graph is the ordered sequence of degrees of the vertices of the graph.
- Exercise: Draw a graph with the following characteristics. If it's not possible, give a reason why not.
  1. A graph on 6 vertices with 12 edges.
  2. A graph with degree sequence $(2, 2, 2, 2, 2, 3, 3, 4)$.
  3. A graph with degree sequence $(1, 2, 3, 4, 5)$.
  4. A graph on 9 vertices with one vertex of degree 4, one vertex of degree 3, and all other vertices having the same degree.

1.4 Some Variations

- Multigraph: allow loops or multiple edges between a pair of vertices.



- Directed Graph: edges are ordered pairs. We say the directed edge (u, v) starts at u and ends at v.

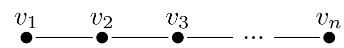


- Graphs can be used to represent many situations. For instance:
  - Flights between cities
  - Friendships between people
  - Shipments between post offices
  - Borders between countries
  - Links between webpages

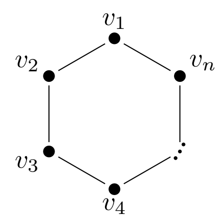
- Discussion: What kind of graph would you use to represent each of the above examples?

1.5 Some Examples

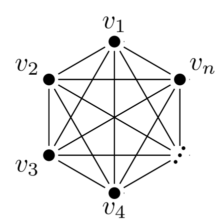
- P_n: path on n vertices



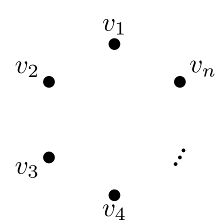
- C_n: cycle of length n



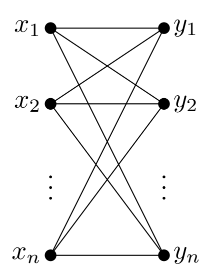
- K_n: complete graph on n vertices



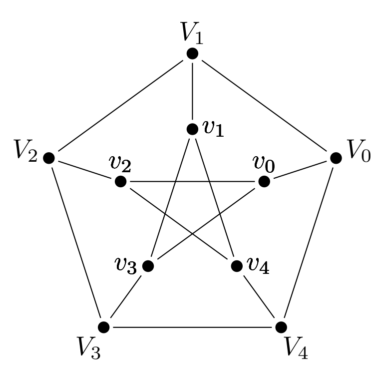
- The empty graph on n vertices



- $K_{n,n}$: complete bipartite graph with parts of size n



- The Petersen graph

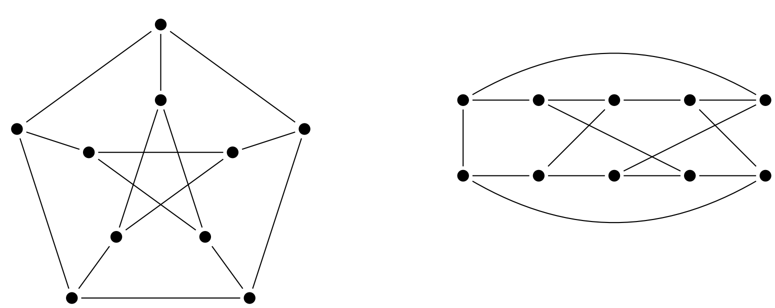


- Exercise: Find as many different graphs on 4 vertices as you can, and identify which ones belong to one of the above families of graphs.

- Discussion: What does it mean for two graphs to be the "same"?

1.6 Isomorphism

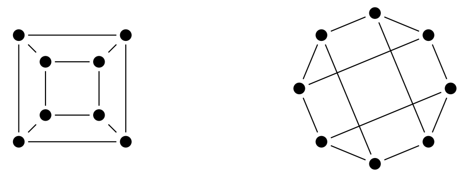
- Two graphs G, H are *isomorphic* if



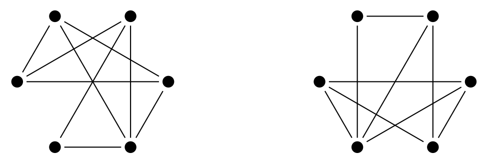
- Exercise: Decide whether the following graphs are isomorphic. If so, give a mapping. If not, give a reason why not.

   1.

   2.



   3.



   4. All graphs with degree sequence $(2, 2, 2, 2, 2, 2)$

- Discussion:

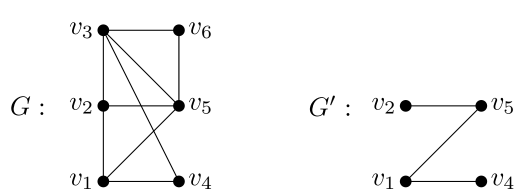
   1. What are some ways to show two graphs are not isomorphic?

   2. Is there any algorithm to decide whether two graphs are isomorphic?

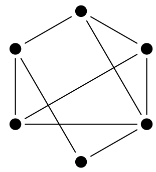
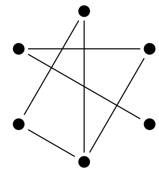
- Note:

1.7 More Definitions

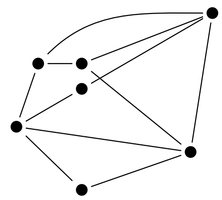
- A graph G' is a *subgraph* of graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$
- A graph G' is an *induced subgraph* of graph G if it is a subgraph and for all $u, v \in V'$, $\{u, v\} \in E(G')$ if and only if $\{u, v\} \in E(G)$.
- Quick check: Is G' a subgraph of G? Is it an induced subgraph?



- The *complement* of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \overline{E(G)}$. That is, $\overline{G}$ has the same vertex set as G, and contains edge $\{u, v\}$ if and only if G does not.

G :  $\overline{G}$: 

- Quick check: Draw the complement of the following graph:

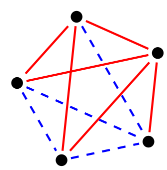
G :  $\overline{G}$:

2 Day 2

2.1 Recap: Handshaking Lemma

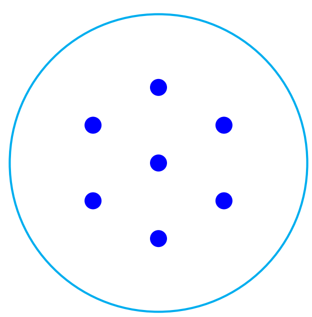
- Lemma: The sum of the degrees of the vertices of a graph is twice the number of edges.
- Exercise: A similar scenario: Suppose ten people are at a party, and each person knows at least five other people there.

  1. Model this situation as a graph. What do you know about the degrees of this graph?
  2. Suppose some people choose to hang out in the garden and the rest hang out outside. If there is always someone inside and someone outside, is it true that there is always someone in the garden who has a friend in the house?
  3. *Bonus:* If you have 5 people at a party, is it possible that there are no 3 people who all know each other and no 3 people who all don't know each other?

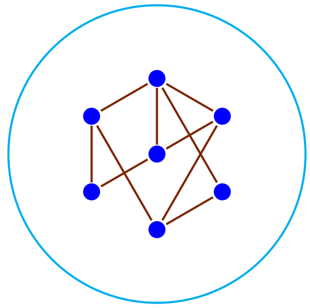
     Equivalently: Can you color the edges of K_5 red and blue so that there is no red or blue triangle? What about K_6? K_7?

2.2 Connecting Islands



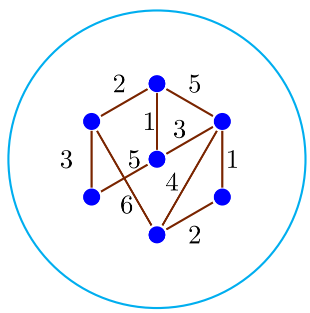
Exercise:

1. What is the minimum number of bridges needed to connect all of the islands? What if there were n islands?



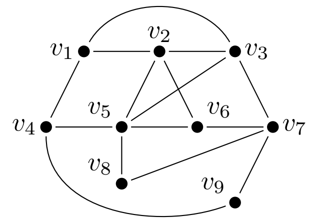
2. What is the minimum number of bridges you can get by removing existing bridges without disconnecting the islands?

3. Would you be able to get down to this number of bridges regardless of the starting configuration of bridges? (Can you draw a counterexample?)



4. *Bonus:* If the bridges had the costs shown above, what is the minimum cost of a set of bridges you'd need to keep in order to keep the islands connected?

2.3 Trees



- A *walk* in a graph is an alternating sequence of vertices and edges, $(v_0, e_1, v_1, e_2, v_2, ..., v_{t-1}, e_t, v_t)$, that starts and ends on vertices such that every edge $e_i = \{v_{i-1}, v_i\}$.
  Note: since we are only discussing simple graphs, we will sometimes denote a walk only by its vertices, $(v_0, ..., v_n)$.
- A *path* is a walk with no repeated vertices.
- A graph $G = (V, E)$ is *connected* if for all $u, v \in V$, there is a path starting from u and ending at v. Such a path is called a uv path.
- Exercise: Show that if there is a walk from u to v in G, then there is a uv path.
- We use the notation $G \setminus v$ (or $G - v$) to denote the graph G with vertex v and all edges incident to v removed. Similarly, $G \setminus e$ (or $G - e$) denotes G with edge e removed.
- Exercise: Let G be the graph above. Find each of the following items or prove they do not exist:
  1. An edge $e \in E(G)$ such that $G - e$ is not connected. Such an edge is called a *cut edge* or a *bridge*.
  2. A vertex $v \in V(G)$ such that $G - v$ is not connected. Such a vertex is called a *cut vertex*.
- A *trail* is a walk with no repeated edges.
- Quick check: Find a trail in the graph in Fig. 3 that is not a path.
- A *circuit* is a trail where the last vertex is the same as the first vertex.
- A *cycle* is a trail where the only repeated vertex is the last vertex.
- Quick Check: Find a circuit in the graph in Fig. 3 that is not a cycle.
- The *length* of a walk is the number of edges it contains.
- The *distance* between two vertices is the length of the shortest path between them.
- Quick check:
  1. What is the distance between v_1 and v_7?
  2. *Bonus:* What is the maximum distance between any two vertices in the graph in Fig. 3?

- A graph is *minimally connected* if it is connected and removing any edge will disconnect it. Such a graph is called a *tree*.

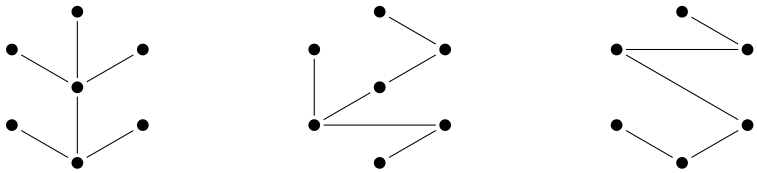
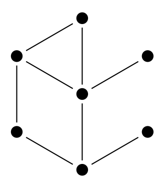


Fig. 4

- Exercise: Draw as many different trees on 6 vertices as you can. Do you notice any patterns?

- Discussion: What are some ways you can tell that a graph is not a tree?



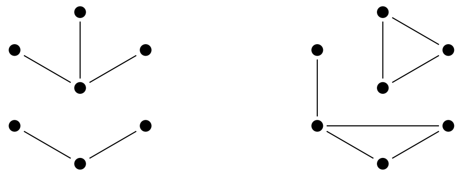
- Exercise: Is a graph without any cycles necessarily a tree?

- Theorem:

  1.

  2.

- If G is not connected, we call the connected "pieces" of G its connected components. More formally, a connected component H of G is a connected subgraph such that for all vertices u in H and v outside of H, there is no uv path.



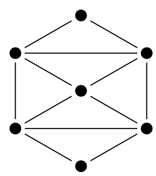
- A *forest* is graph whose connected components are all trees.

2.4 Back to Königsberg

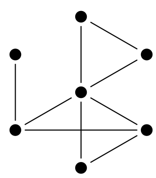
- An *Eulerian circuit* is a circuit which contains every edge of the graph.
- A graph is said to be *Eulerian* if it contains an Eulerian circuit.
- Exercise:

  1. Determine whether following graphs are Eulerian. If yes, write the Eulerian circuit; otherwise, give a reason why not.

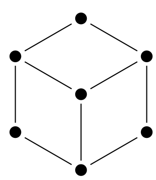
     (a)

     (b)

     (c)

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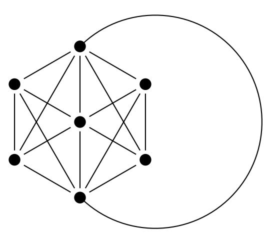
     

  2. Write the degree sequences for each of the graphs. Do you see a pattern?

- (Euler's) Theorem:

- A related notion: a *Hamilton cycle* of G is a cycle that visits every vertex.

- Unlike in the case of Eulerian circuits, there's no known easy way to decide if a graph has a Hamilton cycle.

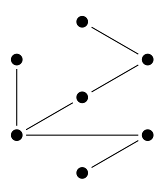
2.5 Induction on Trees

- We previously saw that a tree on n vertices has $n - 1$ edges. In order to formally prove this, let's look a bit further into the structure of a tree.

- A *leaf* in a tree is a vertex of degree 1.

- Exercise: Show that a tree on $n \geq 2$ vertices always has ≥ 2 leaves. (Hint: try to find a long path in the tree.)

- What do we need in order to use induction on trees?



- Theorem: All trees on n vertices

  - We are going to use induction on .

  - Base case:

  - Inductive hypothesis:

  - Inductive statement:

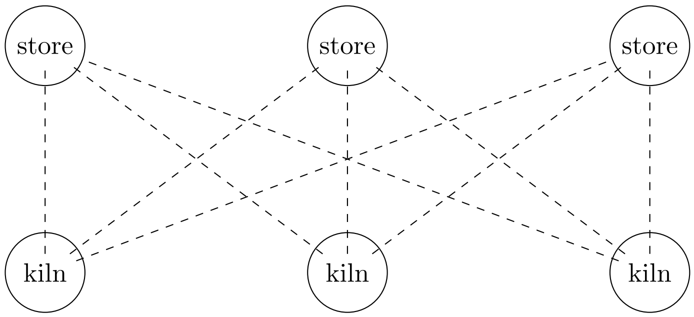
  - Inductive step:

3 Day 3

3.1 Wrap-Up

- Euler's Theorem: HW 2, Problems 5 and 6
- Induction on Trees: Section 2.5

3.2 Turán's Brick Factory Problem



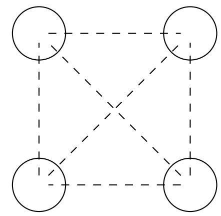


Fig. 2

Exercise: Can Turán avoid track crossings in each of the following situations?

1. There are 2 kilns and 2 storage sites

2. There are 4 buildings with tracks between every pair

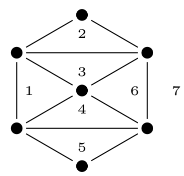
3. There are 3 kilns and 3 storage sites

4. There are 5 buildings with tracks between every pair.

3.3 Planarity

- A *planar* graph is one that can be drawn (on a plane) without any edge crossings.
- Why do we care about planar graphs?
  - Planar graphs represent many concrete situations (circuit boards, utilities lines, tracks).
  - Planar graphs have a lot of structure.
- Discussion: What are some planar graphs we have already seen?

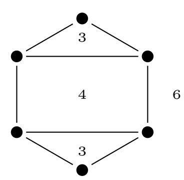
- The vertices and edges of a planar graph divide the plane into distinct regions. These regions are called *faces*.



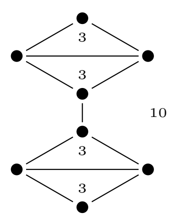
The numbered faces of a planar graph

Note: The region "outside" the graph is also considered a face.

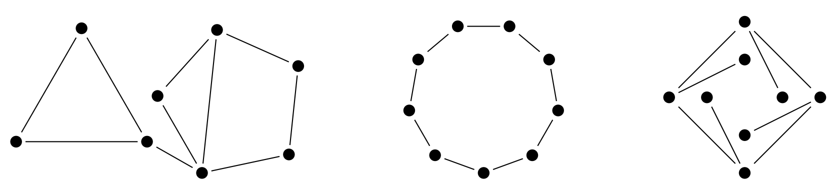
- The *size* of a face is the number of edges bordering it.



The sizes of the faces of a planar graph

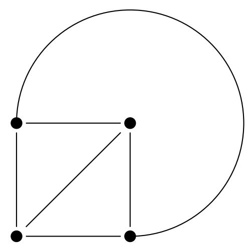


- Exercise: Find the sizes of all of the faces of the graphs below. Can you find a connection to the number of edges?



- Lemma:

- We use $f(G)$ to denote the number of faces of a planar graph G.
- Discussion: Suppose every face in the graph is a triangle. What is the relationship between $e(G)$ (the number of edges in the graph) and $f(G)$ (the number of faces in the graph)?



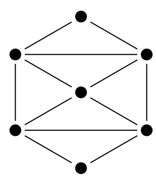
- Discussion: What can we say in general?

3.4 Euler's (Polyhedral) Formula

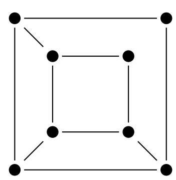
- Exercise:

  1. For each of the following graphs, determine $v(G)$, $e(G)$, and $f(G)$. Do you see a pattern?

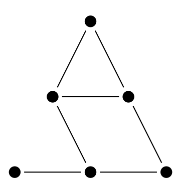
     (a)



     (b)



     (c)



     (d) A cycle on n vertices

     (e) A tree on n vertices

2. How do $v(G), e(G), f(G)$ change if you remove an edge from a cycle? Is the graph still planar? Is the graph still connected?

3. What happens if you keep removing edges from cycles of a planar connected graph until there are no more cycles left?

4. What is the relationship between $v(G), e(G), f(G)$ for a tree?

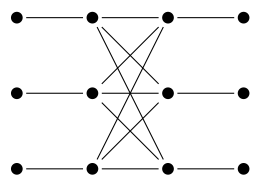
5. What can we conclude about the relationship between $v(G), e(G), f(G)$ for any planar graph?

- Theorem:
  - Note: We will give a proof by induction on

    There are many different proofs (see: https://www.ics.uci.edu/ eppstein/junkyard/euler/)
  - Statement:
  - Base case:
  - Inductive Hypothesis:
  - Inductive Statement:
  - Inductive Step:

3.5 Nonplanar Graphs

- Corollary: If G is planar, connected, and has at least 3 vertices, then $e(G) \leq 3v(G) - 6$
- Exercise: Use the above corollary to prove that K_5 and $K_{3,3}$ are nonplanar
  1. Show that K_5 is not planar.
  2. The *girth* of a graph is the size of the smallest cycle. Can you prove a similar statement to the corollary (giving us a "better" upper bound) when you know the girth of G is g?
  3. Use this to show that $K_{3,3}$ is not planar.
- Discussion: Is the graph below planar? Can we prove it using the corollary?



- Exercise: Draw a nonplanar graph that does not contain K_5 or $K_{3,3}$ as a subgraph
- A graph H is a *subdivision* of graph G if
- Fact: A graph G with subdivision H is planar if and only if H is planar.
- Kuratowski's theorem:

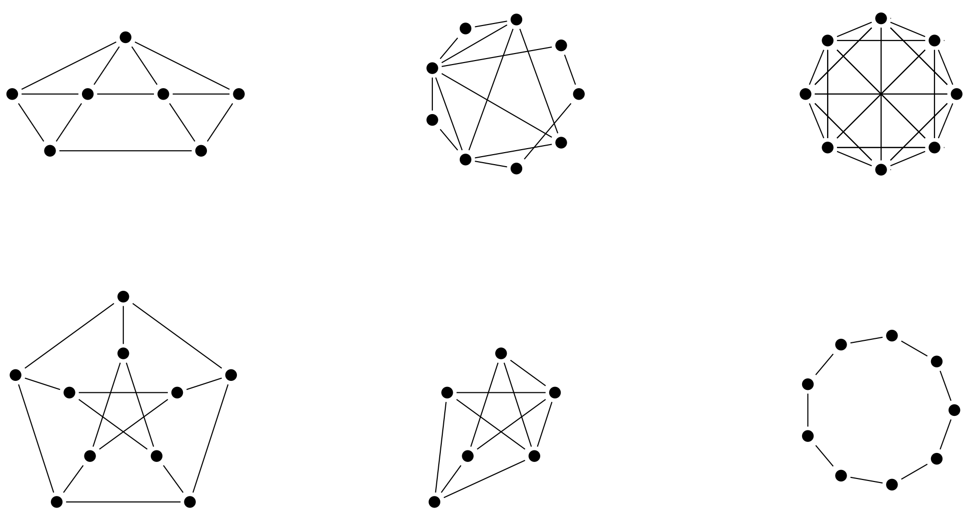
4 Day 4

4.1 Wrap-Up

- Euler's (Polyhedral) Forumula
- Kuratowski's Theorem

4.2 Chromatic Number

- A *(vertex) coloring* of a graph is an assignment of a color to each vertex of the graph. Formally, if C is our set of colors, a coloring of $G = (V, E)$ is a function $f : V \to C$.
- A *proper coloring* is a coloring such that no two adjacent vertices have the same color. Formally, this is a function $f : V \to C$ s.t. $\forall e = \{u, v\} \in E, f(u) \neq f(v)$.
- A graph is called k-colorable if it can be properly colored using at most k colors.
- Applications: Scheduling, frequency assignments, index registers
- Given a color $c \in C$, we call the set of all vertices of colored with c a *color class.*
- Exercise: Construct a proper coloring of the graphs below with as few colors as you can:



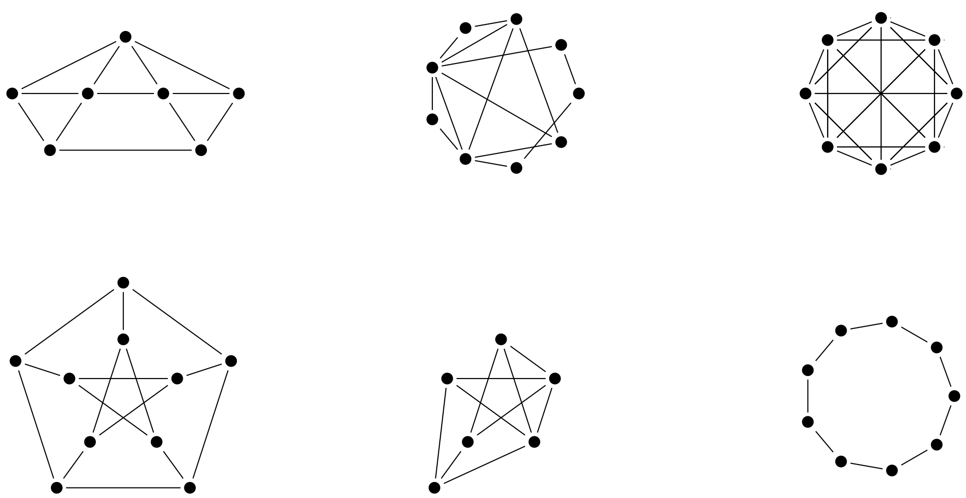
Can you prove you used the smallest possible number of colors?

- The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors needed in a proper coloring of the graph.
- Discussion: What are some ways to *upper bound* $\chi(G)$? What are some upper bounds you can come up with?
- Discussion: What are some ways to *lower bound* $\chi(G)$? What are some lower bounds you can come up with?

- A *clique* is a subset of vertices with edges between every pair. (Sometimes a clique of size k is called a k-clique.)
- The clique number of a graph, $\omega(G)$, is the maximum size of a clique in the graph.
- Fact:
- An upper bound is *tight* is there is ever a case when equality holds.

4.3 Edge coloring

- We can also color the edges of a graph (like we did when trying to find Ramsey numbers).
- An *edge coloring* of a graph is an assignment of a color to each edge in the graph. Formally, if C is our set of colors, an edge coloring of $G = (V, E)$ is a function $f : E \to C$.
- A *proper edge coloring* is a coloring such that no two incident edges have the same color. Formally, this is a function $f : E \to C$ s.t. $\forall e, e' \in E,\ f(e) = f(e') \Rightarrow e \cap e' = \emptyset$.
- A graph is called k-edge-colorable if it can be properly edge colored using at most k colors.
- The *chromatic index* of a graph G, denoted $\chi'(G)$, is the minimum number of colors needed in a proper edge coloring of G.
- Exercise: Construct a proper edge coloring of the graphs below with as few colors as you can:



Can you prove you used the smallest possible number of colors?

- Discussion: What are some ways to *upper bound* $\chi'(G)$? What are some upper bounds you can come up with?
- Discussion: What are some ways to *lower bound* $\chi'(G)$? What are some lower bounds you can come up with?

4.4 2 Small Problems (If Time)

- Maximum $(K_2 \sqcup K_2)$-free graphs
- Optimal random colorings

5 Day 5

5.1 Bounds on chromatic number/chromatic index

- Exercise: What are some bounds we have seen for the chromatic number and chromatic index of a graph? For each of these, try to find an example of a graph where the bound is tight, and try to find an example where it is far from correct.

5.2 Algorithms for Coloring

- An algorithm for coloring a graph:
- This gave us an upper bound on the chromatic number:
- Note:
- Discussion: What is a similar upper bound for the chromatic index?
- Note:
- Random colorings

5.3 Coloring Planar Graphs

- Exercise: Let's consider the previous algorithm more carefully:
  1. How could we improve our bound if we chose an ordering more carefully?
  2. Can we choose a better ordering if we know our graph is planar?

- Theorem:

- The theorem above is very complicated to prove (and is, in fact, one of the first examples of a computer-aided proof). However, we can show a slightly weaker statement.

- Theorem:

- Discussion: Let's modify the previous proof to prove this theorem.

- *Bonus:* before we color the last vertex, can we change the coloring so its neighbors use only 4 colors?

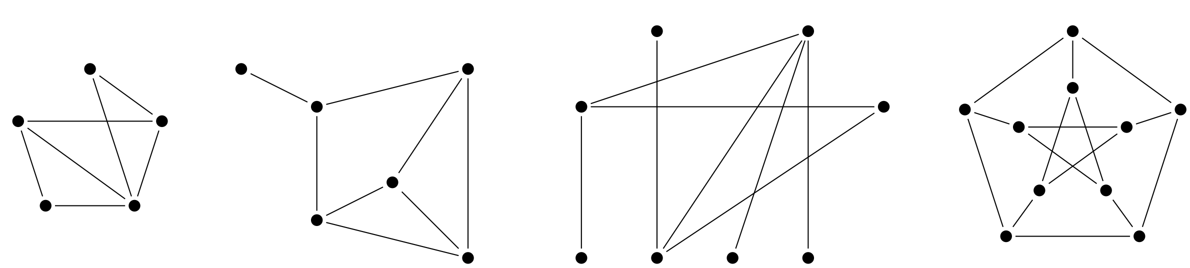
5.4 Matchings

- A *matching* M in a graph G is a subset of edges of G such that no two edges in the subsets are incident to each other. The size of the matching, denoted $|M|$, is the number of edges in the subset.

- There are many applications of matching: scheduling, admissions, organ donation, pairing up students

- A *perfect matching* is a matching where all vertices are incident to an edge in the matching.

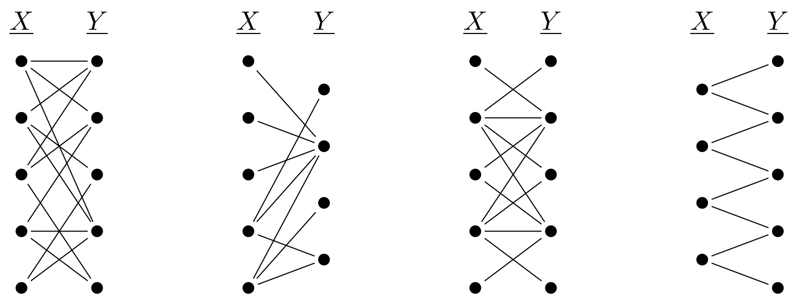
- Exercise: Do the following graphs have perfect matchings? If so, find one; if not, give a reason.



- Discussion: What are some reasons a graph might not have a perfect matching?

- Now let's consider bipartite graphs in particular.
- An *X-perfect matching* of a bipartite graph $G = (X \cup Y, E)$ is a matching where all vertices in X are incident to an edge in the matching.
- Discussion: What are some real-world situations where we only care about X-perfect matchings?

- Exercise: Do the following bipartite graphs have X-perfect matchings? Why or why not?

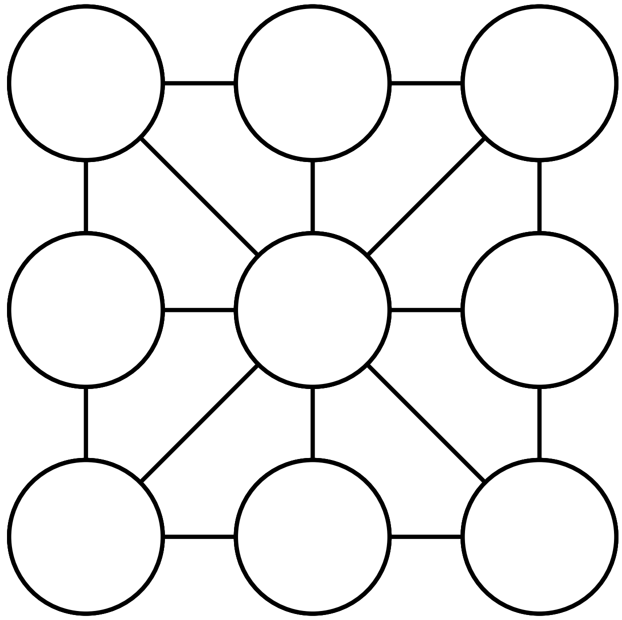


- Discussion: What are some conditions for a graph to have an X-perfect matching?

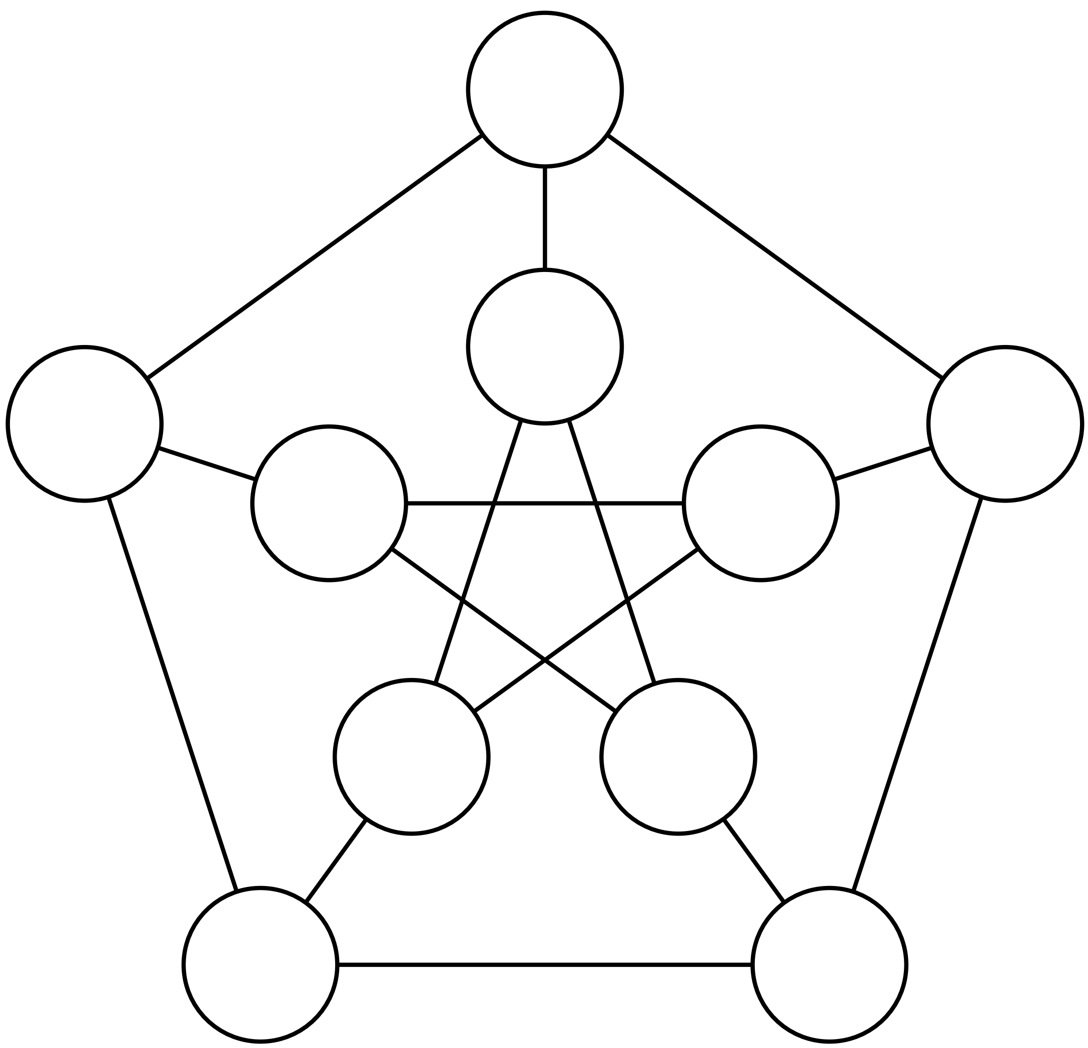
- A *maximum matching* of a graph G is a matching containing the most possible edges.
- The *matching number* of a graph G, denoted $\mu(G)$, is the size of the maximum matching.
- Exercise: Find the matching number of the graphs above. How do you know there can't be a a bigger matching?

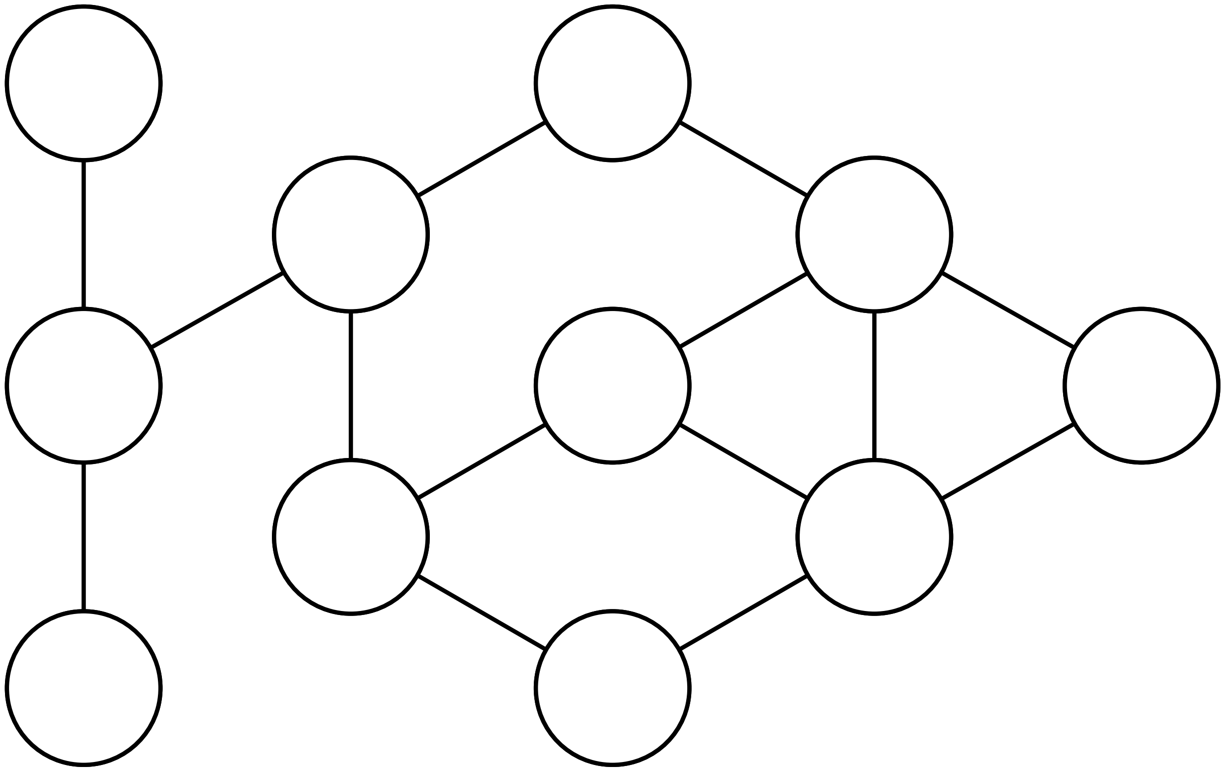
5.5 Games on Graphs

- Achi



- Cops and Robbers





- Try it on your own graphs!