Calculus I Integration: A Very Short Summary

Definition: Antiderivative

The function $F(x)$ is an antiderivative of the function $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I.

Notice, a function may have infinitely many antiderivatives. For example, the function $f(x) = 2x$ has antiderivatives such as x^2, $x^2 + 3$, $x^2 - \pi$, and $x^2 + .002$, just to name a few.

Definition: General Antiderivative

The function $F(x) + C$ is the General Antiderivative of the function $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I and C is an arbitrary constant.

The function $x^2 + C$ where C is an arbitrary constant, is the General Antiderivative of $2x$. This is actually a family of functions, each with its own value of C.

Definition: Indefinite Integral

The Indefinite Integral of $f(x)$ is the General Antiderivative of $f(x)$.

$$\int f(x)\,dx = F(x) + C \qquad\qquad \int 2x\,dx = x^2 + C$$

Definition: Riemann Sum

The Riemann Sum is a sum of the areas of n rectangles formed over n subintervals in $[a, b]$. Here the subintervals are of equal length, but they need not be. The height of the i^{th} rectangle, is the value of $f(x)$ at a chosen sample point in the i^{th} subinterval. The width of each rectangle is $\Delta x = \frac{(b-a)}{n}$ and the height of the rectangle in the i^{th} subinterval is given by $f(x_i^*)$ where x_i^* is a sample point in the i^{th} subinterval. The Riemann Sum used to approximate

$$\int_a^b f(x)\,dx \text{ is given by } \sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Here x_i* is the sample point in the i^{th} subinterval. If the sample points are the midpoints of the subintervals, we call the Riemann Sum the Midpoint Rule.

Definition: Definite Integral

The **Definite integral** of f from a to b, written $\displaystyle\int_a^b f(x)\,dx$, is defined to be the limit of a Riemann sum as $n \to \infty$, if the limit exists (for all choices of sample points $x_1^*, x_2^*, \ldots x_n^*$ in the n subintervals).

Thus, $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \left[\, f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x \,\right]$

The First Fundamental Theorem of Calculus: Let f be continuous on the closed interval $[a, b]$, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f on $[a, b]$.

$$\int_1^3 2x\,dx = x^2\Big]_1^3 = 3^2 - 1^2 = 8$$

The Second Fundamental Theorem of Calculus: Let f be continuous on the closed interval $[a, b]$, and define $G(x) = \displaystyle\int_a^x f(t)dt$ where $a \le x \le b$. Then $G'(x) = \dfrac{d}{dx}\left[\displaystyle\int_a^x f(t)dt\right] = f(x)$.

$$G(x) = \int_0^x \sin^2(t)\,dt \qquad G'(x) = \sin^2(x) \qquad\qquad H(x) = \int_0^{x^3} \sin^2(t)\,dt \qquad H'(x) = 3x^2 \sin^2(x^3)$$

Integration by Substitution: Let $u = g(x)$ and $F(x)$ be the antiderivative of $f(x)$. Then $du = g'(x)dx$ and $\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du = F(u) + C$

Also, $\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du = F\big(g(b)\big) - F\big(g(a)\big)$

Ex: $u = g(x) = x^2$, $\quad du = g'(x)dx = 2x\,dx \qquad \int_2^5 2xe^{x^2}\,dx = \int_{2^2=4}^{5^2=25} e^u\,du = e^u\Big]_4^{25} = e^{25} - e^4$

Integration Rules	**Examples**
$\int k\,du = ku + C$	$\int 3\,du = 3u + C \qquad \int \pi\,dt = \pi t + C$
$\int u^r\,du = \frac{u^{r+1}}{r+1} + C$ for $r \neq -1$	$\int u^5\,du = \frac{u^6}{6} + C$ for $5 \neq -1$
$\int \frac{du}{u} = \int u^{-1}\,du = \ln\lvert u\rvert + C$ for $r = -1$	$\int \frac{2x+3}{x^2+3x}\,dx = \ln\lvert x^2+3x\rvert + \mathrm{C}$
$\int a^u\,du = \frac{a^u}{\ln a} + C$	$\int 4^t\,dt = \frac{4^t}{\ln 4} + C$
$\int e^u\,du = e^u + C$	$\int e^{\sin x}\cos x\,dx = e^{\sin x} + C$
$\int \cos u\,du = \sin u + C$	$\int 3x^2\cos x^3\,dx = \sin x^3 + C$
$\int \sin u\,du = -\cos u + C$	$\int 7t^6\sin t^7\,dt = -\cos t^7 + C$
$\int \sec^2 u\,du = \tan u + C$	$\int 20x^3\sec^2 5x^4\,dx = \tan 5x^4 + C$
$\int \sec u\tan u\,du = \sec u + C$	$\int 4x^3\sec x^4\tan x^4\,dx = \sec x^4 + C$
$\int \csc^2 u\,du = -\cot u + C$	$\int 8\csc^2 8x\,dx = -\cot 8x + C$
$\int \csc u\cot u\,du = -\csc u + C$	$\int 5x^4\csc x^5\cot x^5\,dx = -\csc x^5 + C$
$\int kf(u)\,du = k\int f(u)\,du$	$\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$
$\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$	$\int [4x^3 \pm \sec^2 x]\,dx = x^4 \pm \tan x + C$

Properties of the Definite Integral

$\int_a^a f(x)\,dx = 0$ (same integration limits)

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (exchange integration limits)

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$.

Notice, rules 4 through 6 below are simply negatives of rules 1 through 3.

Inverse Trig Integration Rules	*Examples*
1. $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$	$\displaystyle\int \frac{3x^2\,dx}{\sqrt{1-x^6}} = \sin^{-1} x^3 + C$
2. $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1} u + C$	$\displaystyle\int \frac{4x^3\,dx}{1+x^8} = \tan^{-1} x^4 + C$
3. $\displaystyle\int \frac{du}{\lvert u\rvert\sqrt{u^2-1}} = \sec^{-1} u + C$	$\displaystyle\int \frac{dx}{x\lvert\ln x\rvert\sqrt{(\ln x)^2-1}} = \sec^{-1}(\ln x) + C$
4. $\displaystyle\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1} u + C = -\sin^{-1} u + C$	$\displaystyle\int \frac{-3x^2\,dx}{\sqrt{1-x^6}} = \cos^{-1} x^3 + C = -\sin^{-1} x^3 + C$
5. $\displaystyle\int \frac{-du}{1+u^2} = \cot^{-1} u + C = -\tan^{-1} u + C$	$\displaystyle\int \frac{-4x^3\,dx}{1+x^8} = \cot^{-1} x^4 + C = -\tan^{-1} x^4 + C$
6. $\displaystyle\int \frac{-du}{\lvert u\rvert\sqrt{u^2-1}} = \csc^{-1} u + C = -\sec^{-1} u + C$	$\displaystyle\int \frac{-dx}{x\lvert\ln x\rvert\sqrt{(\ln x)^2-1}} = \csc^{-1}(\ln x) + C = -\sec^{-1}(\ln x) + C$