Well, in my defense...

Pat Devlin
Advisor: Jeff Kahn

Rutgers University

March 27, 2017

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(Especially watching a defense.)

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- Therapy $\rightsquigarrow$ ☺

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Jake Baron
with unidentified stranger

The big picture

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What to expect in this talk?

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What to expect in this talk? Is there food?

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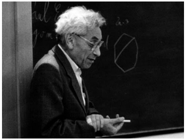
- 34 diet tips to get your beach bod by May! — Buzzfeed *et al.*
  — *You'll never believe number twelve!*

Outline

1. Erdős–Ko–Rado
2. Entropy and Permutations
3. Perfect Fractional Matchings in Hypergraphs
4. Matrices with large permanents

It all started with Erdős



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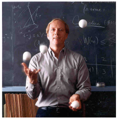




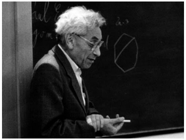


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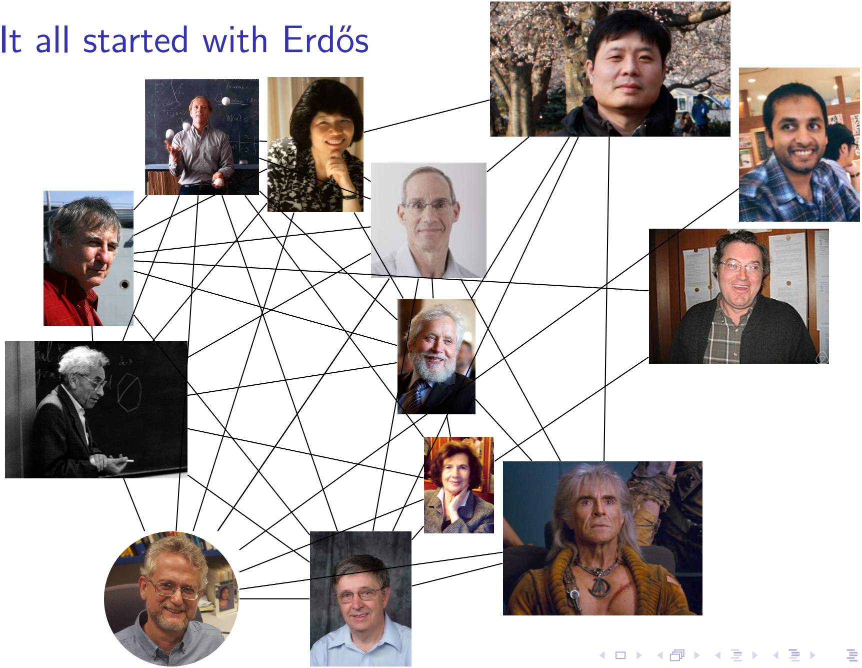

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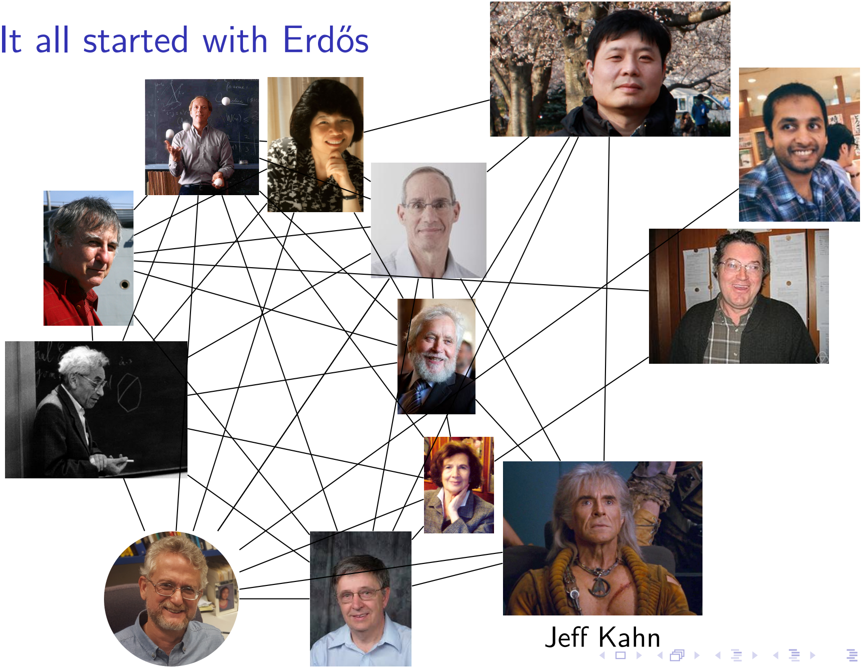

Jeff Kahn

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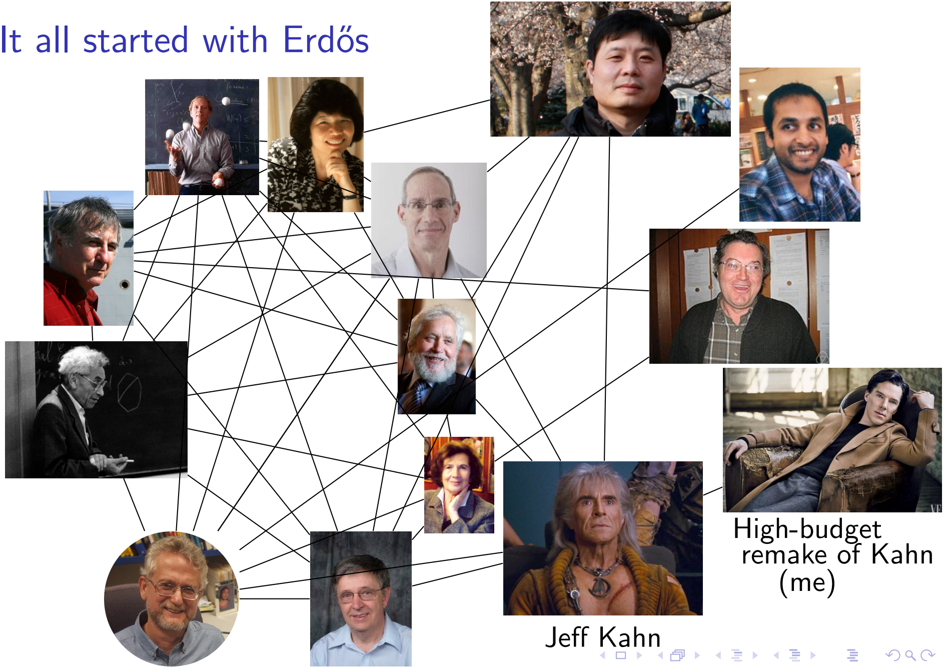

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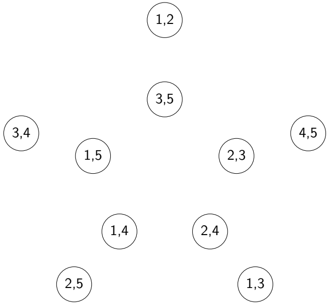

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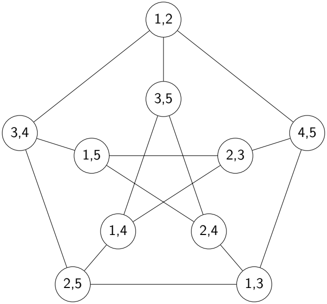

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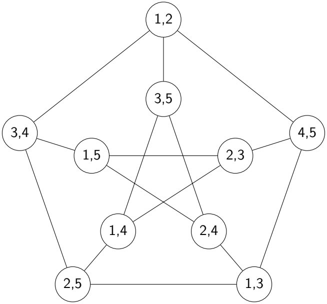

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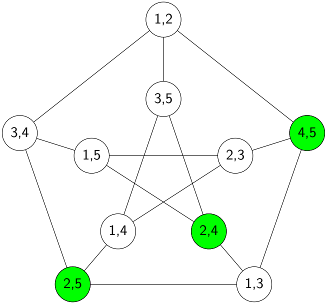

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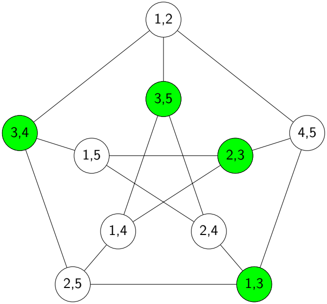

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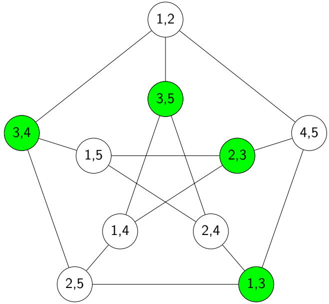

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Any collection of more than $\binom{n-1}{k-1}$ *nodes must contain an edge.*

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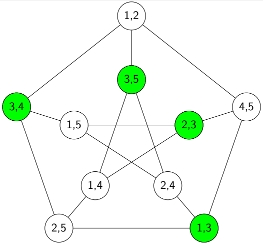

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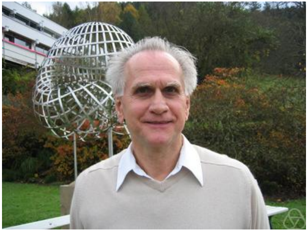

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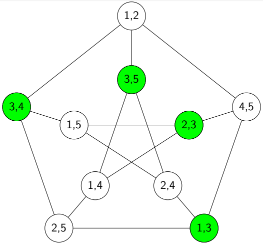

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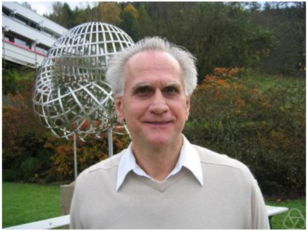

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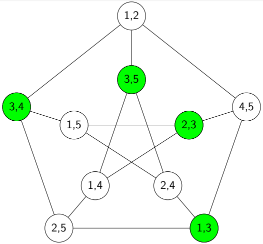

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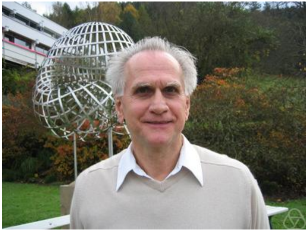

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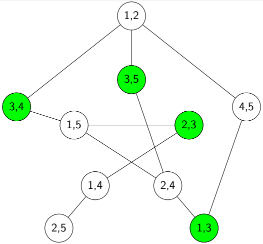

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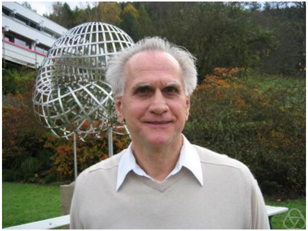

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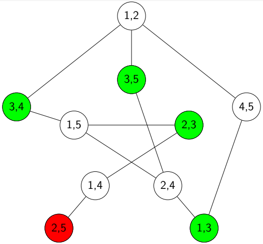

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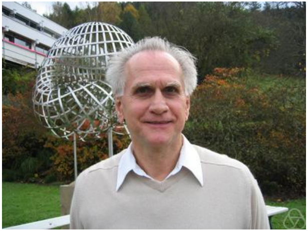

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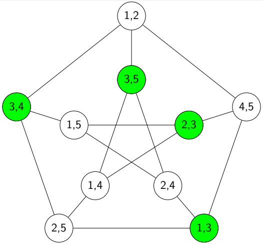

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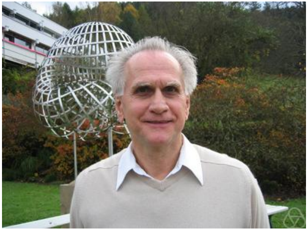

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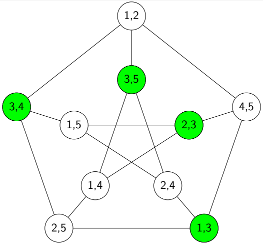

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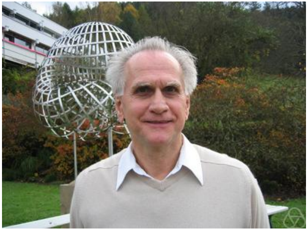

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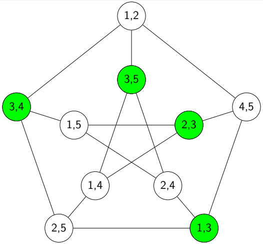

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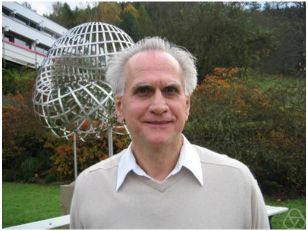

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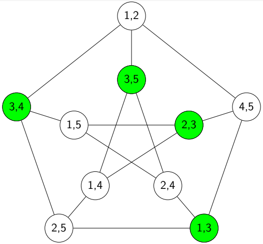

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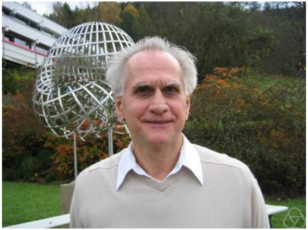

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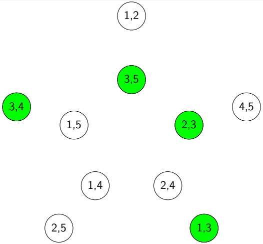

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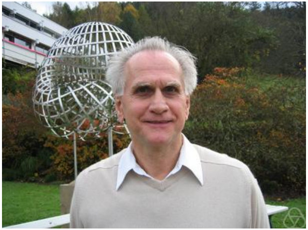

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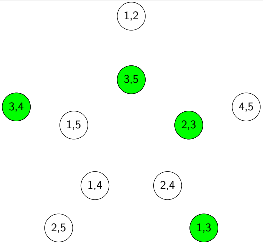

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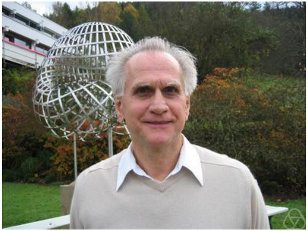

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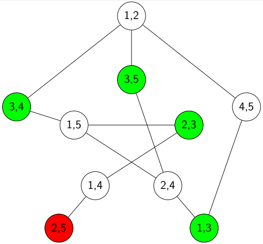

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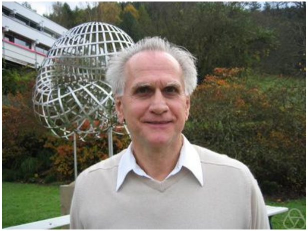

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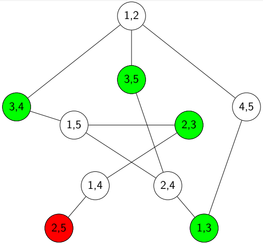

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Theorem (D., Kahn)

If $n > 2k+1$, the threshold probability is $p_c = C\binom{n-k-1}{k-1}^{-1}\log(n\binom{n-1}{k})$. And if $n = 2k+1$, then $3/4 \le p_c < 0.9999999999$.

Outline

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Authors:

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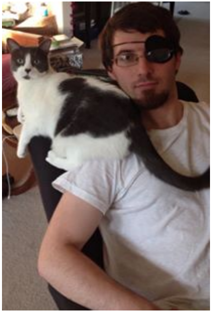

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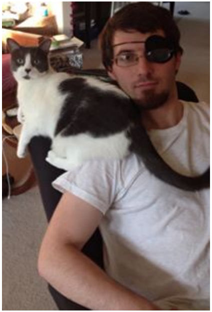
Authors: Hüseyin Acan, me, and Jeff “Shere” Kahn



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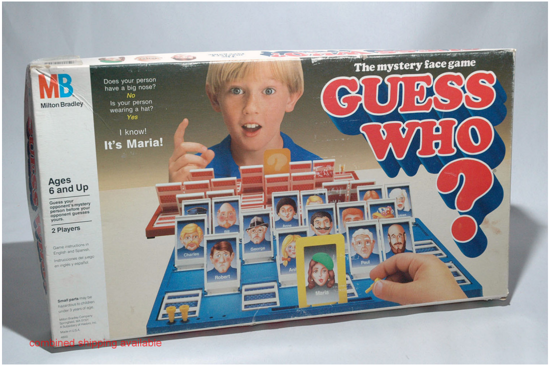

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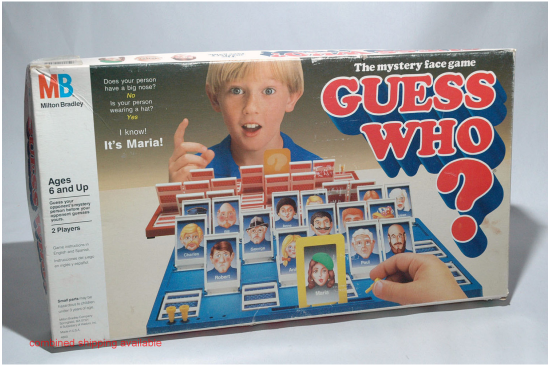

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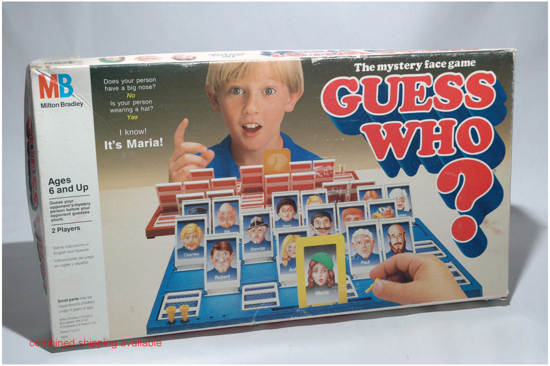

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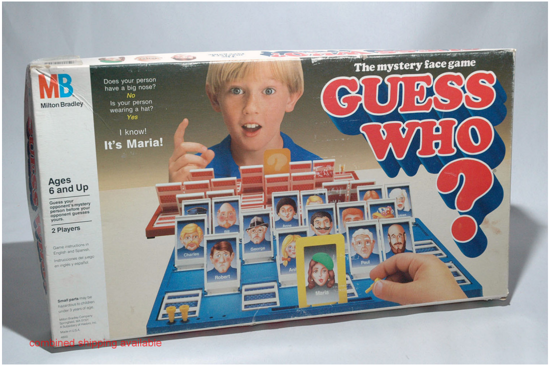
Entropy and permutations: set-up

What is entropy? $H(X) = -\sum_i \mathbb{P}(X=i) \log \mathbb{P}(X=i)$



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If I pick a "random" permutation by a process with noticeable biases, does that mean the outcome is predictable (i.e., has low entropy)?

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We prove $\delta = \exp[-\varepsilon^{-C}]$; it should be ε or ε^2

Outline

Matchings in hypergraphs: submitted (2017+)

Authors: me and Jeff

Matchings in hypergraphs: *very brief*

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Theorem (D., Kahn)

For each r, there is a $k = k(r)$ such that a k-out r-uniform hypergraph has a perfect fractional matching almost surely.

Outline

Matrices with large permanents: to appear ($\approx$ 2016)

Authors:

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Little known fact: these pictures actually depict *different* people.

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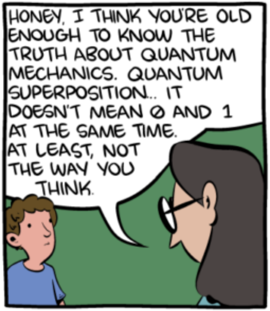
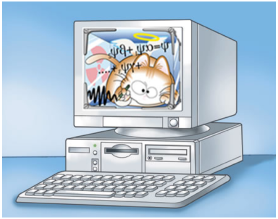
Motivated by connections to quantum computing...





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Theorem (Berkowitz, D.)

If $|\text{perm}(A)| \geq \|A\|^n / n^{100}$, then virtually every row and column of A is dominated by a single entry of large modulus. [A looks very special.]

Matrices with large permanents: results in more detail

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If $\|A\| \leq 1$, then

$$|perm(A)| \leq \exp[-n(1-\delta)^2/10^5].$$

Moreover, if A is over $\mathbb{R}$, then we also have

$$|perm(A)| \leq \exp[-\sqrt{n(1-\delta)}/400].$$

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[Gurvits] $\qquad$ [us]

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This works because of

- intuition/insight. . . ?

Matrices with large permanents: proof sketch

$$\mathrm{perm}(A) := \sum_\sigma \prod_{i=1}^n a_{i,\sigma(i)} \qquad \|A\| := \sup \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$$

$$\text{Goal: } |\mathrm{perm}(A)| \leq \|A\|^n c^{-n}$$

- $|\mathrm{perm}(A)| \leq \mathbb{E}[r.v.]$ (Not unique to us)
- $\mathbb{E}[r.v.] \leq ☺ + \mathbb{P}(r.v. > ☺) \max(r.v.)$
- $\mathbb{P}(r.v. > ☺) \leq ☺$

This works because of

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This works because of

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Nuts and bolts:

- $r.v. \approx \|A\vec{X}\|_1$ $\quad\vec{X} \in \{-1,1\}^n$ uniform
- for $\mathbb{E}[r.v.]$: generalized Khintchine's inequality
- for upper tail: Talagrand, Hoeffding, Bonami hyper-concentration

Matrices with large permanents: proof of Gurvits's result

Let $\vec{X} \in \{-1, 1\}^n$ be chosen uniformly at random. Then

$$\mathrm{perm}(A) = \mathbb{E}\left[\prod_i X_i \left(\sum_{j=1}^n X_j a_{i,j}\right)\right].$$

[Trust me]

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$$\begin{aligned} |\text{perm}(A)| &\leq \mathbb{E}\left[\prod_i |X_i| \left|\sum_{j=1}^n X_j a_{i,j}\right|\right] = \mathbb{E}\left[\prod_i \left|\sum_{j=1}^n X_j a_{i,j}\right|\right] \\ &\leq \mathbb{E}\left[\left(n^{-1} \sum_i \left|\sum_{j=1}^n X_j a_{i,j}\right|\right)^n\right] = \mathbb{E}\left[\left(\frac{\|A\vec{X}\|_1}{n}\right)^n\right] \end{aligned}$$

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\frac{\|A\vec{X}\|_1}{n} &\leq \frac{\|A\vec{X}\|_2\sqrt{n}}{n} = \frac{\|A\vec{X}\|_2}{\sqrt{n}} \leq \frac{\|A\|\|\vec{X}\|_2}{\sqrt{n}} = \|A\|
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\end{aligned}$$

So $|\operatorname{perm}(A)| \leq \|A\|^n$. $\square$

There's no way I need this slide...

Oh no! Stall for more time!

Thanks!

Thanks!

References I

Scott Aaronson and Travis Hance.
Generalizing and derandomizing Gurvits's approximation algorithm for the permanent.
Quantum Inf. Comput., 14(7-8):541–559, 2014.

Scott Aaronson and Hoi Nguyen.
Near invariance of the hypercube.
Israel Journal of Mathematics, 2016.

Hüseyin Acan, Pat Devlin, and Jeff Kahn.
An entropy conjecture of Leighton and Moitra.
2017+.
Submitted. Preprint, arXiv:1701.04321.

References II

Ross Berkowitz and Pat Devlin.
A stability result using the matrix norm to bound the permanent.
Israel Journal of Mathematics, 2016+.
(Accepted, pending publication) Preprint, arXiv:1606.07474.

Béla Bollobás, Bhargav P Narayanan, and Andrei M Raigorodskii.
On the stability of the erdős–ko–rado theorem.
Journal of Combinatorial Theory, Series A, 137:64–78, 2016.

Pat Devlin and Jeff Kahn.
On stability in the Erdős-Ko-Rado Theorem.
SIAM J. Discrete Math., 30(2):1283–1289, 2015.

Pat Devlin and Jeff Kahn.
Perfect fractional matchings for hypergraphs in the k-out model.
2017+.
Submitted. Preprint, arXiv:1703.03513.

References III

Anders Johansson, Jeff Kahn, and Van Vu.
Factors in random graphs.
Random Structures & Algorithms, 33(1):1–28, 2008.

Tom Leighton and Ankur Moitra.
On entropy and extensions of posets, 2011.