Let $A$ be a (finite dimensional, as always) $F$-algebra.

1.1. Let $U$ be an $A$-module. Show that the following statements are equivalent.
   (a) $U$ is completely reducible.
   (b) every submodule of $U$ is completely reducible.
   (c) every homomorphic image of $U$ is completely reducible.

1.2. Let $V$ be an $A$-module. Prove that $V$ is completely reducible iff the intersection of all maximal submodules of $V$ is 0. (Isaacs’ book, problem 1.1).

1.3. $A$ is semisimple iff every $A$-module is completely reducible. (Isaacs’ book, problem 1.3)

1.4. Let $G$ be a finite group and $p$ a prime dividing $|G|$. Let $F$ be a field of characteristic $p$. Prove that $FG$ is not semisimple. (Isaacs’ book, problem 1.9).
   (Hint: Apply Problem 1.2 to the regular module $FG$ and consider the element $\sum_{g \in G} g$.)
2.1. (Isaacs’ book, problem 2.1) a) Let $\Phi$ be an irreducible $\mathbb{F}$-representation of a finite group $G$ over an arbitrary field $\mathbb{F}$. Show that $\sum_{g \in G} \Phi(g) = 0$, unless $\Phi$ is the principal representation.

b) Let $H$ be a subgroup of $G$ and $g \in G$ be such that all elements of the coset $Hg$ are $G$-conjugate. Let $\chi \in \text{Irr}(G)$ be such that $(\chi|_H, 1_H)_H = 0$. Show that $\chi(g) = 0$. (Hint: Apply a) to $H$ and compute $tr(\sum_{h \in H} \Phi(hg))$, $\Phi$ a representation affording $\chi$.)

From now on, all characters are $\mathbb{C}$-characters.

2.2. (Isaacs’ book, problem 2.3) Let $\chi$ be a character of $G$. Choose a representation $\Phi$ affording $\chi$ and define $\det \chi : G \to \mathbb{C}$ as follows: $(\det \chi)(g) = \det(\Phi(g))$. Show that $\det \chi$ does not depend on the choice of $\Phi$, and it is a linear character of $G$.

2.3. (Isaacs’ book, problem 2.4) a) Let $G$ be a non-abelian group of order 8. Show that $G$ has exactly four linear characters and one more irreducible character, say $\chi$, which is of degree 2. Show that $\chi(g) = -2$ if $g \in Z(G)$ and $g \neq 1$, and $\chi(g) = 0$ if $g \in G \setminus Z(G)$. (Hint: Use the fact that $Z(G) = G' \simeq \mathbb{Z}_2$, the cyclic group of order 2, and then compute $(\chi, \chi)_G$.)

b) If $G$ is the dihedral group $D_8$ of order 8:

$$G = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle,$$

then $\det \chi \neq 1_G$. If $G$ is the quaternion group $Q_8$ of order 8:

$$G = \langle a, b \mid a^2 = b^2, a^4 = 1, bab^{-1} = a^{-1} \rangle,$$

then $\det \chi = 1_G$. On the other hand, observe that $D_8$ and $Q_8$ have the same character table. (Hint: Find $\Phi(b)$, $\Phi$ a representation affording $\chi$.)

2.4. (Isaacs’ book, problem 2.13) Let $G$ be a finite group such that $G' \leq Z(G)$ and $|G'| = p$ a prime. Let $\chi \in \text{Irr}(G)$ and $\chi(1) > 1$. Show that $\chi$ vanishes on $G \setminus Z(G)$ and $\chi(1)^2 = (G : Z(G))$.

2.5. (Isaacs’ book, problem 2.9) a) Let $\chi$ be a character of an abelian group $A$. Show that $\sum_{a \in A} |\chi(a)|^2 \geq |A| \cdot \chi(1)$. (Hint: Decompose $\chi|_A$ into irreducible characters of $A$.)

b) Let $G$ be a finite group with an abelian subgroup $A$. Show that $\chi(1) \leq (G : A)$ for any $\chi \in \text{Irr}(G)$. 