Let $A$ be a (finite dimensional, as always) $\mathbb{F}$-algebra.

1.1. Let $U$ be an $A$-module. Show that the following statements are equivalent.
   (a) $U$ is completely reducible.
   (b) every submodule of $U$ is completely reducible.
   (c) every homomorphic image of $U$ is completely reducible.

1.2. Let $V$ be an $A$-module. Prove that $V$ is completely reducible iff the intersection of all maximal submodules of $V$ is $0$. (Isaacs’ book, problem 1.1).

1.3. $A$ is semisimple iff every $A$-module is completely reducible. (Isaacs’ book, problem 1.3)

1.4. Let $G$ be a finite group and $p$ a prime dividing $|G|$. Let $\mathbb{F}$ be a field of characteristic $p$. Prove that $\mathbb{F}G$ is not semisimple. (Isaacs’ book, problem 1.9).
   \textit{(Hint: } Apply Problem 1.2 to the regular module $\mathbb{F}G$ and consider the element $\sum_{g \in G} g$.\textit{)}

2.1. (Isaacs’ book, problem 2.1) a) Let $\Phi$ be an irreducible $F$-representation of a finite group $G$ over an arbitrary field $F$. Show that $\sum_{g \in G} \Phi(g) = 0$, unless $\Phi$ is the principal representation.

b) Let $H$ be a subgroup of $G$ and $g \in G$ be such that all elements of the coset $Hg$ are $G$-conjugate. Let $\chi \in \text{Irr}(G)$ be such that $(\chi|_H, 1_H)_H = 0$. Show that $\chi(g) = 0$. (Hint: Apply a) to $H$ and compute $\text{tr}(\sum_{h \in H} \Phi(hg))$, $\Phi$ a representation affording $\chi$.)

From now on, all characters are $\mathbb{C}$-characters.

2.2. (Isaacs’ book, problem 2.3) Let $\chi$ be a character of $G$. Choose a representation $\Phi$ affording $\chi$ and define $\det \chi : G \rightarrow \mathbb{C}$ as follows: $(\det \chi)(g) = \det(\Phi(g))$. Show that $\det \chi$ does not depend on the choice of $\Phi$, and it is a linear character of $G$.

2.3. (Isaacs’ book, problem 2.4) a) Let $G$ be a non-abelian group of order 8. Show that $G$ has exactly four linear characters and one more irreducible character, say $\chi$, which is of degree 2. Show that $\chi(g) = -2$ if $g \in Z(G)$ and $g \neq 1$, and $\chi(g) = 0$ if $g \in G \setminus Z(G)$. (Hint: Use the fact that $Z(G) = G' \cong \mathbb{Z}_2$, the cyclic group of order 2, and then compute $(\chi, \chi)_G$.)

b) If $G$ is the dihedral group $D_8$ of order 8:
$$G = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle,$$
then $\det \chi \neq 1_G$. If $G$ is the quaternion group $Q_8$ of order 8:
$$G = \langle a, b \mid a^2 = b^2, a^4 = 1, bab^{-1} = a^{-1} \rangle,$$
then $\det \chi = 1_G$. On the other hand, observe that $D_8$ and $Q_8$ have the same character table. (Hint: Find $\Phi(b)$, $\Phi$ a representation affording $\chi$.)

2.4. (Isaacs’ book, problem 2.13) Let $G$ be a finite group such that $G' \leq Z(G)$ and $|G'| = p$ a prime. Let $\chi \in \text{Irr}(G)$ and $\chi(1) > 1$. Show that $\chi$ vanishes on $G \setminus Z(G)$ and $\chi(1)^2 = (G : Z(G))$.

2.5. (Isaacs’ book, problem 2.9) a) Let $\chi$ be a character of an abelian group $A$. Show that $\sum_{a \in A} |\chi(a)|^2 \geq |A| \cdot \chi(1)$. (Hint: Decompose $\chi|_A$ into irreducible characters of $A$.)

b) Let $G$ be a finite group with an abelian subgroup $A$. Show that $\chi(1) \leq (G : A)$ for any $\chi \in \text{Irr}(G)$.
3.1. (Isaacs’s book, problem 3.3) Show that no simple group can have an irreducible character of degree 2. (Hint: Suppose $G$ has such a character of degree 2, afforded by a representation $\Phi$. Then $|G|$ is divisible by 2 and so $G$ has an involution $j$. Use Problem 2.2 to show that $\Phi(j) = -I$, hence $j \in Z(G)$.)

3.2. (Isaacs’s book, problems 3.16 and 3.17) Let $G$ be a finite group of odd order.
   a) Let $\chi \in \text{Irr}(G)$ and suppose $\chi$ is not the principal character. Show that $\chi$ and its complex conjugate $\bar{\chi}$ are different. (Hint: Using the oddness of $|G|$, show that $G = \{1\} \cup \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_m, a_m^{-1}\}$.

   Suppose $\chi = \bar{\chi}$. Computing $(\chi, 1_G)_G$, show that $\chi(1)/2$ is an algebraic integer.)

   b) Suppose $G$ has exactly $r$ conjugacy classes. Show that $|G| \equiv r \pmod{16}$. (Hint: Use part a) and the formula $|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2$.)

3.3. Let $G$ be a non-abelian group of order $p^3$, $p$ a prime. Show that $G$ has exactly $p^2 + p - 1$ irreducible characters: $p^2$ of degree 1 and $p - 1$ of degree $p$. (Hint: First show that $Z(G)$ has order $p$. Then show $[G, G] = Z(G)$ and use Corollary 2.23 and Theorem 3.12.)

   This problem shows that Theorem 3.13 is false if one removes the assumption that $P \in \text{Syl}_p(G)$ is abelian.

3.4. (Isaacs’s book, problems 3.4) Let $G$ be a simple group having an irreducible character $\chi$ of prime degree $p$. Show that $p$, but not $p^2$, divides $|G|$. (Hint: Let $P \in \text{Syl}_p(G)$. If $P$ is abelian, apply Theorem 3.13. If not, show that $\chi|_P$ is irreducible, hence $Z(P) \leq Z(\chi)$ by Schur’s Lemma.)

3.5. Show that any finite non-abelian simple group has order at least 60. (Hint: You can use Burnside’s $p^aq^b$-theorem.)

3.6. Let $g$ be an element of a finite group $G$ and let $k$ be any integer coprime to $|g|$. Show that $g$ is a commutator in $G$ if and only if $g^k$ is a commutator in $G$. (Hint: Use the character formula proved in class.)