Calculus 1000A — Fall 2015
Solutions to Written Assignment 2

Due Date: Oct. 07, 2015 (in class) **Name:** ______________________
Section: 007

- There are two problems in this assignment. Each problem can earn you a maximum of 10 points.
- **Attach extra sheets if necessary.**

Problem 1. State whether the following statements are true or false, and give a brief argument for your answer.

(a) Let $f(x)$ be a continuous function on $[-1,1]$, and $f(-\frac{1}{2}) = 2$, $f(0) = 0$ and $f(\frac{1}{2}) = 1$. Then, f is invertible on $[-1,1]$.

FALSE. As f is continuous on $[-1/2, 0]$, $f(-1/2) = 2$, $f(0) = 0$ and 1 lies between $f(-1/2)$ and $f(0)$, by the Intermediate Value Theorem, there is some number c in $(-1/2, 0)$ such that $f(c) = 1$. But $f(1/2)$ is also 1, so f is NOT ONE-TO-ONE.

(b) If $\lim_{x\to 0} f(x) = 1$, then $\lim_{x\to 0} |f(x)| = 1$.

TRUE. Since $f(x)$ is very close to 1 when x is close to 0, $f(x)$ is positive. So, for x close to 0, $|f(x)| = f(x)$. So, $\lim_{x\to 0} |f(x)| = \lim_{x\to 0} f(x) = 1$.

(c) If $\lim_{x\to 0} |f(x)| = 1$, then $\lim_{x\to 0} f(x) = 1$.

FALSE. Take the example $f(x) = -1$. $\lim_{x\to 0} |f(x)| = 1$, but $\lim_{x\to 0} f(x) = -1$.

(d) The function $f(x) = \log_3(1 - \sin(x))$ is discontinuous at exactly four points. FALSE. $\log_3(x)$ is discontinuous at 0, and $1 - \sin(x)$ takes the value 0 at more than four points, for e.g., at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$ and $\frac{15\pi}{2}$.

Problem 2. Without using the concept of derivatives, compute the following limit:

$$\lim_{x\to\infty} x^2 \cos(\tan^{-1}(4x^2)).$$

(Hint: Using the concepts covered in section 1.5, try to get rid of the trigonometric functions in the above expression.)

Solution. We first simplify the term $\cos(\tan^{-1}(4x^2))$. Let

$$y = \tan^{-1}(4x^2).$$

Then, we know that y is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$\tan(y) = 4x^2.$$

Now, using $\tan^2(y) + 1 = \sec^2(y)$, we see that,

$$\cos^2(y) = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+16x^4}.$$

As $\cos(y)$ is positive for y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have that

$$\cos(\tan^{-1}(4x^2)) = \cos(y) = \frac{1}{\sqrt{1+16x^4}}.$$

Therefore,

$$\begin{aligned}
\lim_{x\to\infty} x^2 \cos(\tan^{-1}(4x^2)) &= \lim_{x\to\infty} \frac{x^2}{\sqrt{1+16x^4}} \\
&= \lim_{x\to\infty} \frac{\frac{x^2}{\sqrt{x^4}}}{\frac{\sqrt{1+16x^4}}{\sqrt{x^4}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^4}+16}} \qquad \text{since } \sqrt{x^4} = x^2 \text{ for all } x \\
&= \mathbf{\frac{1}{4}}.
\end{aligned}$$