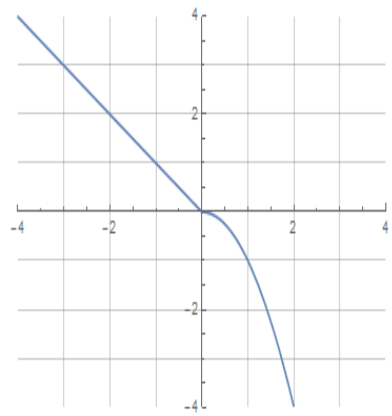
Calculus 1000A — Fall 2015
Solutions to Written Assignment 1

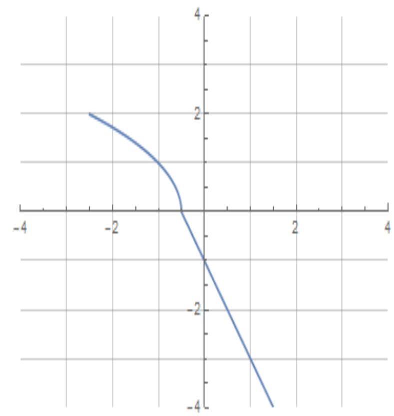
Problem 1. Given below is the graph of the function $f(x)$. $f(x)$ has domain $[-4, 2]$. What is the domain of

$$g(x) = f^{-1}(2x + 1)?$$

In the grid provided below, carefully sketch the graph of $g(x)$.



graph of $f(x)$



graph of $g(x)$

Domain of $g(x)$: $[-\frac{5}{2}, \frac{3}{2}]$.

(Useful tip: First, determine the domain and draw the graph of $h(x) = f^{-1}(x)$.)

Problem 2. (*i*) Use the addition and subtraction formulas for the sine and cosine functions to prove the identity

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

(*ii*) Using part (*i*), find all the pairs (x, y), $0 \le x, y < \frac{\pi}{2}$, that satisfy **both** of the following equations:

$$2\sin(x-y) = \sin(2x) - \sin(2y); \quad (1)$$

$$x = 4y. \quad (2)$$

(Useful tip: Simplify (1) before substituting (2) in (1).)

Solution. (*i*) Using the addition formula for cos and the subtraction formula for sin, we can write

$$\cos\left(\frac{x+y}{2}\right) = \cos\left(\frac{x}{2}+\frac{y}{2}\right) = \cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right);$$

$$\sin\left(\frac{x-y}{2}\right) = \sin\left(\frac{x}{2}-\frac{y}{2}\right) = \sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) - \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right).$$

Multiplying the two equations, we get

$$\begin{aligned}
&\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\\
= &\left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)\left(\sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) - \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)\\
= &\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right) - \cos^2\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)\\
&\quad - \sin^2\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right) + \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right)\\
= &\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\left(\cos^2\left(\frac{y}{2}\right)+\sin^2\left(\frac{y}{2}\right)\right) - \cos\left(\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)\left(\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)\right).
\end{aligned}$$

Now, $\cos^2(x) + \sin^2(x) = 1$ and $\sin(2x) = 2\sin(x)\cos(x)$, so we can write

$$2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \sin(x) - \sin(y).$$

(*ii*) We use the identity proved in (*i*) to re-write the left-hand side of (1), and obtain:

$$2\sin(x-y) = 2\cos\left(\frac{2x+2y}{2}\right)\sin\left(\frac{2x-2y}{2}\right) = 2\cos(x+y)\sin(x-y).$$

Thus, (1) can be re-written as $2\sin(x-y)(1-\cos(x+y)) = 0$.

Case *a*. $\sin(x-y) = 0$. Subsituting $x = 4y$ in this equation, we get $\sin(3y) = 0$. For y in $[0, \frac{\pi}{2})$, this happens at $y = 0$ and $y = \frac{\pi}{3}$. The corresponding values of x are 0 and $\frac{4\pi}{3}$, respectively. But, $\frac{4\pi}{3}$ is not in the interval $[0, \frac{\pi}{2})$. So, $(0, 0)$ is a solution.

Case *b*. $1 - \cos(x+y) = 0$. Subsituting $x = 4y$ in this equation, we get $\cos(5y) = 1$. For y in $[0, \frac{\pi}{2})$, this happens at $y = 0$ and $y = \frac{2\pi}{5}$. The corresponding values of x are 0 and $\frac{4\pi}{5}$, respectively. But, $\frac{4\pi}{5}$ is not in the interval $[0, \frac{\pi}{2})$. So, **$(0,0)$ is the only solution**.