6.2. Volumes.

Dec. 07, '15

Recall: Volume of solids of revolution:

$$V = \int_a^b \pi\left((r_{out})^2 - (r_{in})^2\right) dx$$

(about the x-axis)

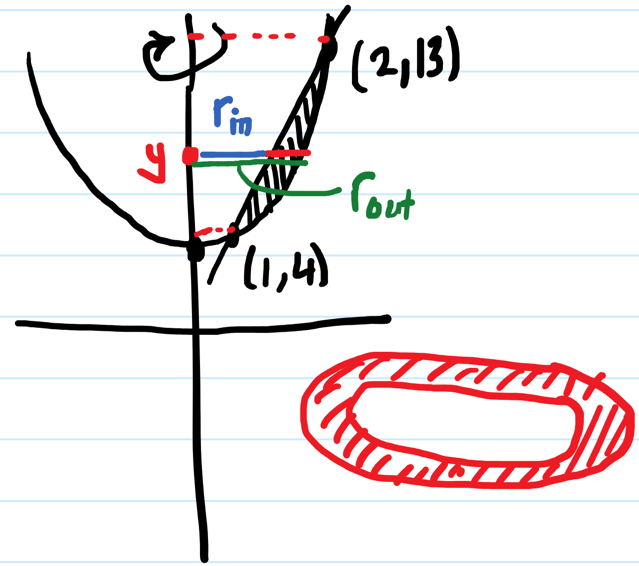
Region: bounded by $y = 3x^2+1$ and $y = 9x-5$.

$x=1$

Points of intersection.

$$3x^2+1 = 9x-5$$
$$3x^2-9x+6=0$$
$$x^2-3x+2=0$$
$$x^2-2x-x+2=0$$
$$(x-1)(x-2)=0.$$

So, the two curves intersect at $(1,4)$ and $(2,13)$.



(1) About the y-axis.

$$y = 3x^2+1 \rightarrow x = \sqrt{\frac{y-1}{3}}$$

$$y = 9x-5 \rightarrow x = \frac{y+5}{9}$$

$$r_{in} = \frac{y+5}{9}, \quad r_{out} = \sqrt{\frac{y-1}{3}}$$

$$V = \int_4^{13} \pi\left(\sqrt{\frac{y-1}{3}}\right)^2 - \pi\left(\frac{y+5}{9}\right)^2 dy$$

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$$= \pi\int_4^{13} \frac{y-1}{3}\,dy - \pi\int_4^{13}\left(\frac{y+5}{9}\right)^2 dy$$

$u = \frac{y-1}{3}$

$du = \frac{1}{3}dy$

$\underline{3du = dy}$

$y=4 \rightarrow u = \frac{4-1}{3} = 1$

$y=13 \rightarrow u = \frac{13-1}{3} = 4$

$u = \frac{y+5}{9}$

$du = \frac{1}{9}dy$

$\underline{9du = dy}$

$y=4 \rightarrow u = \frac{4+5}{9} = 1$

$y=13 \rightarrow u = \frac{13+5}{9} = 2$

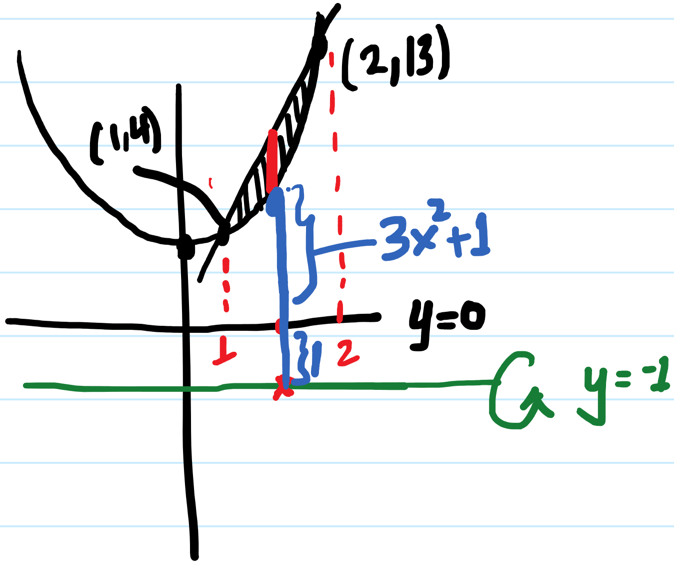
$$= \pi\int_1^4 u(3du) - \pi\int_1^2 u^2 \cdot 9du$$

$$= 3\pi\left[\frac{u^2}{2}\Big|_1^4\right] - 9\pi\left[\frac{u^3}{3}\Big|_1^2\right]$$

$$= 3\pi\left[\frac{16-1}{2}\right] - \overset{3}{\cancel{9}}\pi\left[\frac{8-1}{\cancel{3}}\right]$$

$$= \frac{45\pi}{2} - 21\pi = \frac{(45-42)\pi}{2}$$

$$= \frac{3\pi}{2}.$$



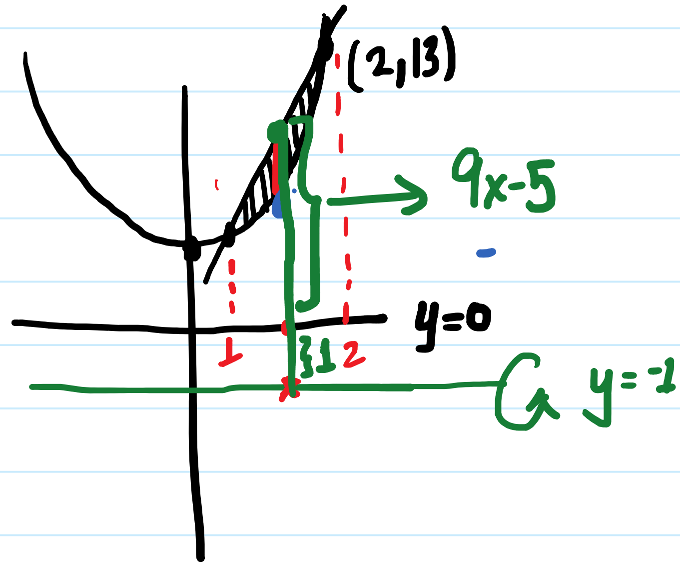
$y = 3x^2+1, \quad y = 9x-5.$

Revolve about $y=-1$. (parallel to the y-axis – so leave in terms of x)

$r_{in} = 3x^2 + \underbrace{1+1}$ → axis of revolution is one unit away from the x-axis.

$= 3x^2+2$

$r_{out} = 9x-5+1 = 9x-4$

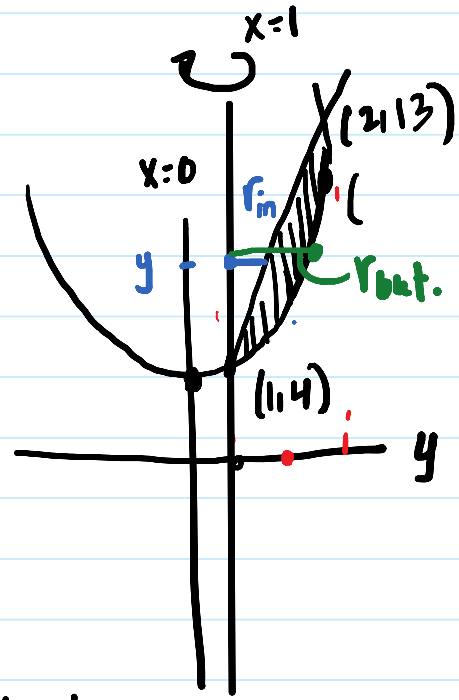


$$V = \int_1^2 \pi\left[9x-4\right]^2 - \pi\left[3x^2+2\right]^2 dx$$

HW

↳ Compute this.

[If the axis was $y=1$, we would subtract 1 from r_{in} and r_{out}.]



Revolve about x=1.

Parallel to the y-axis - so setup everything in terms of y.

$$x = \sqrt{\frac{y-1}{3}} \quad , \quad x = \frac{y+5}{9}.$$

$$r_{in} = \frac{y+5}{9} - 1.$$

$$r_{out} = \sqrt{\frac{y-1}{3}} - 1.$$

$$V = \int_4^{13} \pi \left(\sqrt{\frac{y-1}{3}} - 1 \right)^2 - \pi \left(\frac{y+5}{9} - 1 \right)^2 dy.$$

↳ Compute! HW

QUIZ 4 Solutions

1. $$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{2}{n}\,\overbrace{\ln\left(1+\frac{4i^2}{n^2}\right)}^{\ln\left(1+\left(\frac{2i}{n}\right)^2\right)}\right) =$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\,f\left(a+i\frac{(b-a)}{n}\right).$$

easy $\rightarrow$

$\leftarrow$ hard

	a	b	$a+i\left(\frac{b-a}{n}\right)$	$f(x)$	$f\left(a+i\left(\frac{b-a}{n}\right)\right)$
(A) $\int_1^3 \ln(1+x^2)\,dx$	1	3	$1+i\frac{2}{n}$	$\ln(1+x^2)$	$\ln\left(1+\left(1+\frac{i2}{n}\right)^2\right)$
(B) $\int_0^2 \ln(x^2)\,dx$	0	2	$\frac{i2}{n}$	$\ln(x^2)$	$\ln\left(\left(\frac{i2}{n}\right)^2\right)$
(C) $\int_0^2 \ln(1+x^2)\,dx$	0	2	$\frac{i2}{n}$	$\ln(1+x^2)$	$\ln\left(1+\left(\frac{i2}{n}\right)^2\right)$ ✓
(D) $\int_1^3 \ln(x^2)\,dx$	1	3	$1+i\frac{(2)}{n}$	$\ln(x^2)$	$\ln\left(\left(1+\frac{i2}{n}\right)^2\right).$

(E) 0.

2. $\dfrac{d}{dx}\displaystyle\int_{a(x)=0}^{b(x)=e^x} \underbrace{\frac{u^2-1}{u^2+1}}_{f(u)}\,du$

$$= \frac{(e^x)^2-1}{(e^x)^2+1}\cdot e^x - \cancel{\frac{0-1}{0+1}\cdot 0}$$

$$= e^x\,\frac{e^{2x}-1}{e^{2x}+1}. \qquad (B).$$

$$\to \boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(u)\,du = f(b(x))\,b'(x) - f(a(x))\cdot a'(x).}$$

3. $\displaystyle\int \boxed{\frac{\ln(\sin x)\cos x}{\sin x}}\,dx$ ✱

$u=\sin x \qquad du=\cos x\,dx$

$$= \int \frac{\ln(u)}{u}\,du \qquad V=\ln(u) \quad dV=\frac{1}{u}\,du$$

$$= \int V\,dV = \frac{V^2}{2}+C = \frac{\ln(u)^2}{2}+C$$

D $\qquad = \dfrac{(\ln(\sin x))^2}{2}+C.$

OR differentiate each solution to get ✱

4. $\displaystyle\int_{-3}^{3} \frac{1}{1+x^4} - \sin^3 x \, dx$

$$= \int_{-3}^{3} \underbrace{\frac{1}{1+x^4}}_{\text{even}} dx - \underbrace{\int_{-3}^{3} \underbrace{\sin^3 x}_{\text{odd.}} dx}$$

$$= 2\int_{0}^{3} \frac{1}{1+x^4} dx - 0$$

$$= 2\int_{0}^{3} \underline{\frac{1}{1+x^4}} dx \quad A.$$

$$\sin^3(-x) = (-\sin x)^3$$

$$= -(\sin x)^3$$

$$= -\sin^3 x$$

Exam Review

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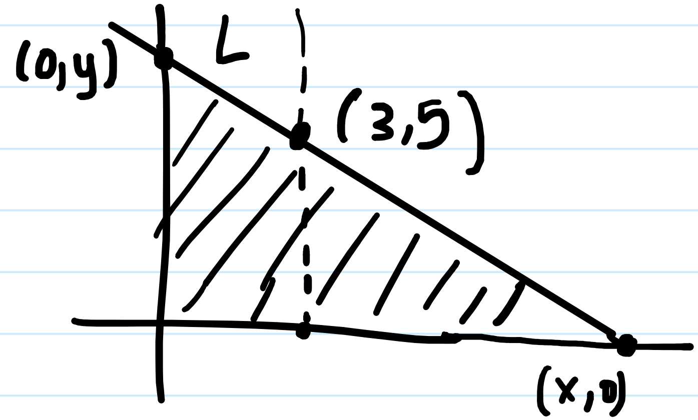
Final Exam 2013

Find an equation of the line through $(3,5)$ that cuts off the least area from the first quadrant.



To minimize : Area of the Triangle $= \frac{1}{2} \cdot x \cdot y$

Slope of L : $\frac{5-0}{3-x} = \frac{y-5}{0-3} \to \frac{5}{3-x} = \frac{5-y}{3}$

$\to \frac{15}{3-x} = 5-y$

$\to y = 5 - \frac{15}{3-x} = \frac{\cancel{15}-5x\cancel{-15}}{3-x} = \frac{-5x}{3-x}$.

To minimize : $A(x) = \frac{1}{2}x\left(\frac{-5x}{3-x}\right) = \frac{5x^2}{2(x-3)}$. in $(3,\infty)$

$A'(x) = \frac{5}{2}\left[\frac{(x-3)(2x)-x^2(1)}{(x-3)^2}\right] = \frac{5}{2}\left(\frac{2x^2-6x-x^2}{(x-3)^2}\right) = \frac{5}{2}\,\frac{x(x-6)}{(x-3)^2} \to A'(x)=0$

$\cancel{x=0}$, $x=6$

3 — 4 (<0) — 6 — 7 (>0)

$A'(4) = \frac{5}{2}\cdot\frac{4(-2)}{1^2} < 0$, $A'(7) = \frac{5}{2}\cdot\frac{7\cdot 1}{(4)^2} > 0$] FDT.

To minimize: $A(x) = \frac{1}{2}x\left(\frac{-5x}{3-x}\right) = \frac{5x^2}{2(x-3)}$ in $(3,\infty)$

$$A'(x) = \frac{5}{2}\left[\frac{(x-3)(2x) - x^2(1)}{(x-3)^2}\right] = \frac{5}{2}\left(\frac{2x^2-6x-x^2}{(x-3)^2}\right) = \frac{5}{2}\,\frac{x(x-6)}{(x-3)^2} \rightarrow A'(x) = \;\; \cancel{x=0} \;\; x=6$$

3 $\quad <0 \quad >0$

$\left[A'(4) = \frac{5}{2}\cdot\frac{4(-2)}{2} < 0, \quad A'(7) = \frac{5}{2}\cdot\frac{7\cdot 1}{2} > 0 \right]$

By the First Derivative Test $A(x)$ has a global minimum at $x=6$. in $(3,\infty)$.

Slope of the line: $\frac{5-0}{3-6} = \frac{5}{-3} = -\frac{5}{3}$

The line that cuts off the minimum area in the first quadrant

is $\boxed{(y-0) = -\frac{5}{3}(x-6)}$

35.

Calculate

$$\int_0^a x\sqrt{\underbrace{a^2-x^2}_{u}}\,dx$$

$u = a^2 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$x=0 \to u = a^2 - 0^2 = a^2$

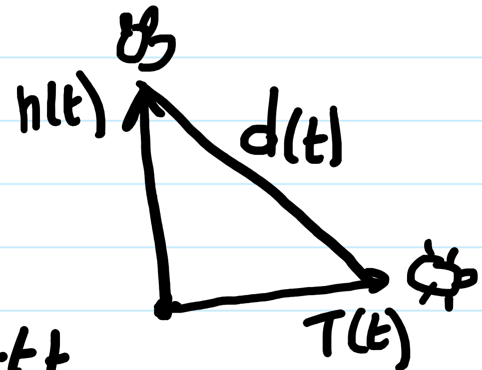
$x = a \to u = a^2 - a^2 = 0.$

$$= \int_{a^2}^{0} \sqrt{u}\; -\frac{1}{2}du = -\frac{1}{2}\int_{a^2}^{0} \sqrt{u}\,du = -\frac{1}{2}\left(\frac{u^{3/2}}{3/2} \Big|_{a^2}^{0} \right)$$

$$= -\frac{1}{\cancel{2}}\left(-\frac{\cancel{2}}{3}(a^2)^{3/2} \right)$$

$$= \underline{\underline{\frac{1}{3}a^3}}.$$

36. A hare and a tortoise start moving from the same point. The hare travels north at 1.5 km/hour and the tortoise travels east at 2 km/hour. At what rate is the distance between them increasing 2 hours later?



To find $d'(2)$,

$$d(t)^2 = h(t)^2 + T(t)^2$$

Know: $h'(t) = 1.5$, $T'(t) = 2$

diff. w.r.t t

$$2 \cdot d(t)\, d'(t) = 2h(t)\, h'(t) + 2T(t)\, T'(t)$$

$$2d(2)\, d'(2) = 2h(2)(1.5) + 2T(2)2$$

$$2 \cdot 5\, d'(2) = 2 \cdot 3 \cdot \overset{9}{(1.5)} + 2 \cdot 4 \cdot 2$$

$$d'(2) = \frac{25}{10} = 5/2 \text{ km/hour.}$$

$h(2) = 1.5 \times 2$ km $= 3$

$T(2) = 2 \times 2 = 4$ km

$d(2) = \sqrt{3^2 + 4^2} = 5.$

The distance between the hare and the tortoise increasing at $5/2$ km/hr after 2 hours.

The distance between the hare and the tortoise was after 2 hours.